Welfare Comparison of Allocation Mechanisms under Incomplete Information

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Abstract

We study the problem of allocating n objects to n agents without monetary transfers in a setting where each agent's preference is *privately* known. We show that when each agent's ranking over objects is independent of other agents' rankings and each possible ranking is equally likely, the celebrated Random Serial Dictatorship mechanism is unambiguously welfare *inferior* to another allocation method, the Random Boston mechanism, when the number of agents and objects is large. More precisely, every type of every agent has a higher interim utility under the Random Boston mechanism. This result also has an implication about the welfare comparison of two widely used allocation methods for school choice, the Deferred Acceptance (DA) mechanism and the Boston mechanism: When each school ranks students identically, the Boston mechanism is welfare superior to the DA mechanism in the same strong manner in large markets.

Keywords: Allocation without transfers; Bayesian Incentive Compatibility; Random Serial Dictatorship; School Choice; Deferred Acceptance mechanism; Boston mechanism

1 Introduction

It is impossible or sometimes simply impractical to use monetary transfers in many real-life allocation problems. On-campus housing assignment, office allocation among faculty members, and student placement into public schools are notable examples among many others. In this paper, we consider the problem of allocating n indivisible objects to n agents,

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where each agent is entitled to receive exactly one object, when monetary transfers cannot be used. We consider a setting in which there is *incomplete information* about agents' preferences and compare allocation methods in terms of welfare of agents.

The Random Serial Dictatorship (RSD) mechanism (sometimes referred to as the Random Priority (RP) mechanism) is one of the most popular allocation methods. Under RSD, first an order over agents is drawn uniformly at random and, by following this order, the first agent is assigned his most preferred object, the second agent is assigned his most preferred object among the remaining ones, and so on. RSD is a straightforward method with its transparent and easy to implement rules. More importantly, RSD has strong incentive properties. More precisely, RSD is strategy-proof, that is, truth-telling is a weakly dominant strategy. Furthermore, RSD yields the same random allocation with the core (competitive allocation) from random endowments—endowment matchings drawn from uniform distribution. (See Abdulkadiroğlu and Sönmez (1998).) These support the wide use of RSD in practical applications. On the other hand, Bogomolnaia and Moulin (BM) (2001) present an example with four objects and four agents with a set of preference profiles in which another random allocation is unambiguously better than RSD for each agent. Manea (2009) shows that instances of such inefficiency of RSD are in fact prevalent in large allocation problems. In particular, he shows that as the number of objects increases, the fraction of preference profiles for which this kind of inefficiency does not arise vanishes. Our main contribution is to exhibit an allocation method that is welfare superior to RSD in a setting in which each agent has a strict preference ranking which is *privately* known.[1]

We consider another allocation method, the Random Boston (RB) mechanism. We begin by informally describing the workings of this method for which a more detailed and formal explanation will be given later in Section 3. Under RB, each agent reports a strict ranking over objects and in the first step, each object is allocated to an agent who ranks it as a *first* choice, if there is any such agent. If there are more than one agent who ranks an object as a first choice, a random draw among these agents determines who gets that object. In the second step, each available object is allocated to an agent who ranks it as a *second* choice among those who didn't receive an object in the first step, if there is any such agent, and again randomly when necessary. We move on in this fashion until all the objects are allocated. As an example, suppose three objects, $\{o_1, o_2, o_3\}$, will be allocated to three agents, $\{i_1, i_2, i_3\}$.

[1] Contrary to the most of the literature, in our setting agents don't know others' preferences, which is more realistic in many instances. We should also note here that there are other papers in the literature that consider an incomplete information setting. (See Roth and Rothblum(1999) and Ehlers (2008), for example, that use the same informational setting as ours.) There are relatively recent papers on the school choice problem that considers incomplete information setting. We will discuss those later in the Introduction.

Assume that agents' reported preference rankings are as follows. Agent i_1 ranks objects as $o_1 \succ o_2 \succ o_3$, agent i_2 ranks as $o_1 \succ o_3 \succ o_2$, and agent i_3 ranks as $o_3 \succ o_2 \succ o_1$. Consider object o_1. Both i_1 and i_2 rank it as a first choice. Hence, object o_1 will be given to either i_1 or i_2 by a fair coin toss. Without loss, let's say agent i_1 receives o_1. There is no one who ranks object o_2 as a first choice and i_3 is the only agent who ranks object o_3 as a first choice, so i_3 gets object o_3. Thus, object o_2 remains unassigned in the first step and is carried to the next step. In the second step, the only available object is o_2, and it will be assigned to agent i_2, who didn't receive an object in the first step and who ranks o_2 as a second choice. Thus, all the objects are allocated, and each agent receives one object.[2]

We show that RB is welfare superior to RSD in a very strong manner in large allocation problems in a setting where each agent's ranking over objects is independent of others' rankings and drawn from a uniform distribution over the set of all possible rankings. Our main result—precise statement is presented as Theorem 1, below—is:

> *When the number of objects and agents is sufficiently large, every type of every agent has a strictly higher interim utility under the Random Boston mechanism than under the Random Serial Dictatorship mechanism.*[3]

To prove the result, we first establish an asymptotic interim (first-order) stochastic dominance relation. Precisely, we show that for each $K \geq 1$, the interim probability of obtaining one of first K choices under RB is strictly greater than that under RSD when n, the number of objects and agents, is large enough. We then show that this asymptotic stochastic dominance easily implies that for sufficiently large n, every type of every agent has a higher interim utility under RB regardless of the preference intensities. Hence, strong incentive properties of RSD comes with a huge welfare cost even compare to another allocation method.

As noted above, the welfare deficiency of RSD has also been demonstrated earlier by Bogomolnaia and Moulin (BM) (2001). Noting the inefficiency of RSD, BM define a new allocation method, the Probabilistic Serial (PS) Mechanism, which is based on an algorithm in which agents "eat" probability shares of available objects and show that PS is ordinally

[2] As its name suggests, the Random Boston mechanism is a simple variant of the Boston mechanism, a widely-used allocation method in the school choice context, which will be explained later, with the difference that objects don't have preferences/priorities over agents—objects are indifferent among agents— and hence possible ties among agents are resolved randomly and uniformly.

[3] The *interim* stage refers to the stage in which each agent knows his/her own preference but not others' preferences. Hence, the interim utility is the expected utility, where expectation is over the preferences of other agents.

efficient in that it cannot be (first-order) stochastically dominated by another random allocation.[4] Che and Kojima (2010) show that in a large economy with many copies of objects, PS and RSD are in fact asymptotically equivalent. Hence, the inefficiency of RSD compare to PS becomes small in large markets. Note that our result not only exhibits the welfare deficiency of RSD but also explicitly finds a mechanism that *dominates* RSD. Furthermore, our result is obtained in a setting where there is incomplete information about preferences.

Our result about the object allocation problem where there is only one-sided preference also has an implication about a debated comparison in the school choice context where there are two-sided preferences.[5] There are two widely used mechanisms for assigning students to schools: the well-known Deferred Acceptance (DA) mechanism[6] and the Boston mechanism. Our main result for object allocation problem implies the following welfare comparison result for these school choice mechanisms.

Assume that each student's ranking over schools is independent of others' rankings and each possible strict ranking is equally likely. Furthermore, assume that all schools have identical priorities over students, say, because they all rank students according to some test score,[7] and each possible ranking over students is equally likely. Then, when each school has one available seat, our main result implies:

> *When the number of students and schools is sufficiently large, every type of every student has a strictly higher interim utility under the Boston mechanism than under the Deferred Acceptance mechanism.*

The Deferred Acceptance and the Boston mechanisms have been the predominant allocation methods in many school districts in U.S. Given the widespread use, there have been a huge literature analyzing these mechanisms and trying to figure out which one is "better". Earlier work compared these two school choice mechanisms mostly in terms of their incentive properties. The DA mechanism has been favored on these grounds since truthful reporting of preferences by students is a weakly dominant strategy under the DA. (See Dubins and Friedman (1981), Roth (1982).) The Boston mechanism, on the other hand, is known to be vulnerable to strategic manipulation. Students may gain from misreporting their prefer-

[4] Note that this result does *not* imply that PS dominates RSD. Rather, it says random allocation that is induced by PS cannot be dominated by any other random allocation including the allocation induced by RSD.

[5] Students have preferences over schools, and schools have priorities/preferences over students.

[6] Student Proposing Gale-Shapley Algorithm (Gale&Shapley(1962)).

[7] We elaborate more on this assumption later in this section.

ences under the Boston mechanism (See Abdulkadiroğlu and Sönmez (2003), for example.).[8] Furthermore, when school preferences are strict, the DA mechanism produces the student optimal stable outcome. On the other hand, as Ergin and Sönmez (2006) show, any stable matching can be obtained in a Nash equilibrium of the Boston mechanism under *complete* information.

The welfare comparison of these two school choice mechanisms, until very recently, has been studied under complete information when student preferences and school priorities are commonly known. Ergin and Sönmez (2006) show that the DA mechanism is welfare superior to the Boston mechanism under complete information.[9] Again, in a complete information environment, Chen and Sönmez (2006) provide experimental evidence that DA is better in terms of welfare. There is a recent line of literature that compares these mechanisms under incomplete information. Abdulkadiroğlu, Che and Yasuda (2011) (henceforth, ACY) compare these two mechanisms in terms of welfare under incomplete information but in a very special case. They show that when students' ordinal rankings are *identical*,[10] and schools have no priorities, the Boston mechanism interim dominates the Deferred Acceptance mechanism, that is, each student's interim payoff is weakly higher under the Boston mechanism in this special case. Troyan (2012), allowing for more general priority structures for schools but still assuming that students' ordinal preferences are identical, shows that the Boston mechanism is ex-ante welfare superior to the DA mechanism. In another work, Featherstone and Niederle (2013) provide experimental evidence that the Boston mechanism may dominate the DA mechanism in terms of ex-ante welfare and, in addition, truth-telling rates are almost the same when the preferences are uncorrelated. When there is a continuum of students, Miralles (2009) shows that the DA mechanism performs poorly when all students have the same ordinal rankings over schools.

Here we step away from the restrictive assumption that students have the same ordinal rankings over schools. Instead, we consider a polar opposite case: students' preferences are independently distributed and each ordinal ranking over schools is ex-ante equally likely. Under this assumption, the problem becomes significantly different from the ones where each student's ranking over schools is identical. For example, when all students have the same ranking and schools have no priorities over students, as in ACY, the DA mechanism becomes

[8] In our setting with symmetric and independent preferences, however, the Boston mechanism is (Bayesian) incentive compatible, as will be discussed.

[9] They also show via an example that when students have private information about their preferences, this result does not hold and some type of students have higher utility under the Boston mechanism.

[10] That is, each student's ranking over schools is the same. For example, school s_1 is the first choice of *every* student, school s_2 is the second choice of *every* student and so on.

a trivial random assignment. More precisely, each student is assigned to each school with equal probabilities. This is a crucial simplification which enables the welfare comparison result in ACY. However, when students' ordinal rankings may differ from one another as in this paper, the equivalence of DA to random assignment no longer holds, and analysis of the aforementioned mechanisms is much more involved. Nevertheless, our symmetric setting allows us to do explicit welfare calculations which otherwise seem intractable. Although the symmetric setting in which each ranking is ex-ante equally likely is another extreme case, we believe that it is still important to understand how these two mechanisms perform in terms of welfare when students' preferences may differ and to the best our knowledge ours is the first attempt in this direction. Furthermore, our result can be applied to the following case as well. Suppose there are groups of schools. Say, all students rank some schools higher than others but students' preferences *within* each group can differ from each other, say, due to location preferences, and are uncorrelated. Suppose that the system is such that students first apply to schools in the higher ranked group and those who are not able to get into this group of school apply to schools in the second best group and so on.[11] Then, the same comparison result between Boston and DA still holds.

We now elaborate more on the assumption that schools have *identical* preferences (or, priorities) over students. In many countries—for instance, China, Iran, Japan and Turkey—students take centralized tests and the scores on these are the sole determinant of school priorities. In other systems, schools do not have a preference over students per se, and a central lottery is used to assign priorities to students. The manner in which schools' preferences are determined has implications about what students know when they submit their preferences. In the test score interpretation, it is reasonable to assume that each student knows how he or she is ranked by the schools when submitting preferences. In the lottery interpretation, however, a student does not know his or her priority when submitting preferences. Our comparison result, however, is not sensitive to the two specifications—the result that the Boston mechanism yields a higher *interim* welfare than the DA mechanism holds in *both* cases.[12]

The rest of the paper is organized as follows. We first present the environment and

[11] An example of such a situation had been in action in Turkey. Students were taking a centralized exam to be able to get into the "Science High Schools"— commonly believed to be the best high schools— and those who couldn't get into these schools were getting into other high schools for some of which there were exams again. Later, this system has been abondened and currently one centralized exam is being carried out for all types of high schools.

[12] From now on, unless otherwise stated, we will focus on the lottery interpretation so that students don't know the priorities when reporting preferences.

the main result for the object allocation problem. Then, we formally describe the above-mentioned student assignment mechanisms, the Boston mechanism and the DA mechanism, and state the result about the welfare comparison of these. Then, we conclude. The proofs that are missing in the main body are relegated to the Appendix.

2 Object Allocation Problem

An object allocation problem is a one in which a number of objects will be allocated to a number of agents. An **object allocation mechanism** is a procedure that assigns objects to agents, where each agent receives one object, given agent's strict rankings over objects. We will implicitly consider allocation mechanisms to be direct mechanisms that ask students to report rank orderings over objects and implement the corresponding procedure given these reports.

We now describe two allocation mechanisms, namely the Random Serial Dictatorship mechanism and the Random Boston mechanism.

First, each agent reports a strict ranking over objects (which, of course, may or may not be the true ranking) and given the reports, these mechanisms allocate objects to agents as follows.

The Random Serial Dictatorship mechanism

Given a strict order over agents, $\pi = (\pi_1, ..., \pi_n)$, where π_i denotes the agent in the i^{th} rank of this order,

Step 1*:* Agent π_1 is assigned his favorite object.

For $k \geq 2$,

Step k: Agent π_k is assigned his favorite object among the ones that haven't been assigned in Step 1,..., Step $(k-1)$.

This method is called The Serial Dictatorship Mechanism. When the order over agents is determined by a random draw from a uniform distribution over the set of all agents, it is called The Random Serial Dictatorship Mechanism.

The Random Boston mechanism

Step 1. For each object, consider agents who rank this object as a *first* choice, if any. If there is exactly one such agent, assign the object to this agent. If the number of such agents is more than one, determine the agent who gets the object according to a random draw among these agents. If there is an object that hasn't been allocated, or an agent who hasn't received an object, then move to Step 2. Otherwise, stop.

For $k \geq 2$,

Step k. For each object that hasn't been allocated in earlier step(s), consider agents who rank this object as a k^{th} choice among those who haven't obtained an object in earlier step(s). Again, allocate objects among these agents, and randomly when necessary. If there is an object that hasn't been allocated, or an agent who hasn't received an object, then move to Step $(k+1)$. Otherwise, stop.

Note that the algorithm will stop at step $k = n$ or before.

3 The Model

Assume that there are n agents, $\mathcal{I} = \{i_1, ..., i_n\}$, and n objects, $\mathcal{O} = \{o_1, ..., o_n\}$, $n \geq 2$. The allocation mechanisms we consider require agents to report rank orderings (that is, strict preferences). Therefore, we assume that each agent has a strict ranking over objects[13] which is *privately* known. We assume that each agent's ranking over objects is independent of other agents' rankings and each possible ranking is equally likely. Each agent obtains nonnegative payoff from consuming each object.

3.1 Incentive Compatibility

We first investigate agents' incentives for truthful reporting under these mechanisms.

A mechanism is (Bayesian) incentive compatible if truthful reporting of preference rankings is a Bayes-Nash equilibrium.[14]

[13]Possibly, except the ones he obtains 0 utility from consuming, as will be discussed below.

[14]That is, an agent should report a higher rank for an object than another if he prefers former strictly more than the latter.

First, it is easy to see that, and as is well known, the Random Serial Dictatorship mechanism is strategy-proof, that is, truthful reporting of preferences is a weakly dominant strategy,[15] and hence (Bayesian) incentive compatible. On the other hand, in general, truthful reporting may not be an equilibrium under the Random Boston mechanism. We now present an example to illustrate how an agent can become better off by manipulating his preference list under RB.

Example 1 *There are three agents $\{i_1, i_2, i_3\}$ and three objects $\{o_1, o_2, o_3\}$. Assume that each agent's type(ranking over objects) is independently drawn from the following distribution.*

$$\begin{aligned} \Pr(o_1 \succ o_2 \succ o_3) &= p = \frac{3}{4} \\ \Pr(o_2 \succ o_1 \succ o_3) &= 1 - p = \frac{1}{4} \end{aligned}$$

Without loss, let's consider agent i_1. Assume that i_1 derives a utility of 1 from his top choice, utility of 0.8 from his second choice and utility of 0 from his third choice.
Assume that i_2 and i_3 report truthfully and let's check whether it is a best response for i_1 to report truthfully.
Consider i_1 with type $o_1 \succ o_2 \succ o_3$. If he reports truthfully, that is, if he reports his ranking as $o_1 \succ o_2 \succ o_3$, his interim payoff will be

$$p^2\left(\frac{1}{3}(1) + \frac{1}{3}(0.8) + \frac{1}{3}(0)\right) + 2p(1-p)\left(\frac{1}{2}(1) + \frac{1}{2}(0)\right) + (1-p)^2(1) = \frac{47}{80}$$

This is due to the following observations. With probability p^2 both of the other agents' preferences are also $o_1 \succ o_2 \succ o_3$, and hence agent i_1 will receive each object with probability $\frac{1}{3}$. With probability $2p(1-p)$, one of the other agent's ranking is $o_1 \succ o_2 \succ o_3$ and the other's preference ranking is $o_2 \succ o_1 \succ o_3$. In this case, agent i_1 will obtain his first choice o_1 with probability $\frac{1}{2}$. If he doesn't receive o_1, he will receive o_3 since one of the agents rank o_2 as a first choice, so o_2 will be assigned to that agent in Step 1 and thus object o_3 is left to agent i_1. Finally, with probability $(1-p)^2$, both of the other agents rank objects as $o_2 \succ o_1 \succ o_3$. In this case agent i_1 will receive o_1 in Step 1 since he is the only one who ranks this object as a first choice.

If agent i_1 reports his preference as $o_2 \succ o_1 \succ o_3$, by similar arguments his interim payoff

[15] See Abdulkadiroğlu and Sönmez (1998), for example.

will be

$$p^2\,(0.8) + 2p\,(1-p)\left(\frac{1}{2}\,(0.8) + \frac{1}{2}\,(0)\right) + (1-p)^2\left(\frac{1}{3}\,(1) + \frac{1}{3}\,(0.8) + \frac{1}{3}\,(0)\right) = \frac{51}{80}$$

Hence, agent i_1 *with type* $(1, 0.8, 0)$ *can gain from misreporting his preferences.*

However, truthful reporting of preferences becomes an equilibrium under the Random Boston mechanism when preferences are symmetric and independent:

Proposition 1 *Assume that each agent's preference ranking over objects is independent of other agents' rankings and each possible strict ranking is equally likely. Then, truth-telling is an equilibrium under the Random Boston mechanism.*

Proof. See Appendix A.1. ■

Let's consider the above example to get an idea why truth-telling becomes an equilibrium in the setting here. Note that in that example the rankings over objects are not equally likely and agent i_1 with type $(o_1 \succ o_2 \succ o_3)$ knows that other agents' rankings over objects are *highly likely* to be $o_1 \succ o_2 \succ o_3$, which is also his true ranking. Hence, agent i_1 with type $(o_1 \succ o_2 \succ o_3)$ can avoid this highly likely tie by reporting $o_2 \succ o_1 \succ o_3$, which allows him to obtain his second choice for sure. And, when his utility from his second choice is not much lower than his utility from his top choice, as in the above example, he becomes better off. However, under symmetry assumptions, this kind of situation does not arise since every possible report of other agents is equally likely, which leads agents to report truthfully.

3.2 Welfare Comparison

Given these observations about agents' incentives under the Random Boston mechanism and the Random Serial Dictatorship mechanism, we compare these mechanisms in terms of welfare of agents under truth-telling.[16]

Mechanism A *(strictly) interim dominates* mechanism B if every type of every agent has a (strictly) higher interim welfare under mechanism A than under mechanism B regardless of the preference intensities, that is, the interim utility of every type of every agent is (strictly) higher under mechanism A.[17]

[16] Importantly, we don't impose truth-telling. It is an equilibrium in our setting.

[17] Recall that the "interim stage" refers to the stage when each agent *only* knows his or her own type and the interim utility is the expected utility calculated at the interim stage.

Denote $(P_k^n)^{RB}$ as the interim probability that an agent obtains his k^{th} choice under the Random Boston mechanism when there are in total n agents and n objects and similarly denote $(P_k^n)^{RSD}$ as the interim probability that an agent obtains his k^{th} choice under RSD when the number of agents and objects is n.[18]

We first establish an asymptotic interim (first-order) stochastic dominance relation between two allocation mechanisms. The proof is relegated to the Appendix.

Lemma 1 *For all $K \geq 1$,*

$$\lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^{RB} > \lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^{RSD}$$

In words, for any $K \geq 1$, the interim probability of obtaining one of first K choices under RB is strictly greater than that under RSD when the number of objects and agents is large enough.

In fact, the interim probability of obtaining the *first* choice is always higher under RB. It is easy to see this even without doing any computation.[19] Under RB, an agent, say agent i_1, compete for his top choice, say object o_1, only with agents who also rank object o_1 as a first choice. Thus, if he is the luckiest among all such agents, that is, if he ranks above all such agents in the random draw at the first step of RB, he gets his first choice. On the other hand, under RSD, even if, as a result of the random draw, agent i_1 ranks above all the agents who rank object o_1 as a first choice, it may be the case that another agent, say agent i_2, whose second choice is object o_1 may get it before it's agent i_1's turn since agent i_2's first choice may have been assigned to an agent who ranks above i_2 in the random order.

Although there is no such clear intuition to see the relation between the interim probabilities of obtaining one of first K choices for a general $K \geq 2$, we show in the Appendix by explicit calculations that the interim probability is higher under RB for any K for large enough n.[20]

We are now ready to state our main result. We will assume that there is a number $M \geq 1$, which can be as large as possible, such that the number of desirable objects is at most M for each agent. That is, each agent derives a positive utility from at most M objects and a

[18] Note that these probabilities don't depend on the identity of agents or labels of objects since these mechanisms treat identical agents/objects identically.

[19] For a formal proof of this claim, we refer the reader to the Appendix.

[20] Computer simulations show that for $K = 2$, we need $n > 4$, for $K = 3$, we need $n > 6$, for $K = 4$, we need $n > 9$, and so on.

utility of 0 from the rest of the objects.[21]

Theorem 1 *Assume that there is some number $M \geq 1$ such that the number of desirable objects for each agent is at most M. Furthermore, assume that each agent's ranking over objects is independent of other agents' rankings and each possible strict ranking over the set of desirable objects is equally likely. For sufficiently large n, the Random Boston mechanism strictly interim dominates the Random Serial Dictatorship mechanism.*

Proof. The proof easily follows from the above Lemma. We refer the reader to the Appendix for details. ■

Hence, better incentive properties of RSD comes with a huge welfare cost. There is another mechanism that is better for any agent regardless of the preference intensities.

We next look at the problem of assigning students to schools and compare two widely used school choice mechanisms, the Deferred Acceptance mechanism and the Boston mechanism. By observing that in a special setting, these mechanisms are equivalent to the object allocation mechanisms studied in this section. Therefore, our main result regarding the object allocation mechanisms implies a similar welfare comparison result between these two school choice mechanisms.

4 Implications about the School Choice Problem

Let's consider the school choice problem, the problem of assigning students to schools. Different from assigning objects to agents, there are two-sided preferences in the school choice problem. Students have preferences over schools, and schools have preferences/priorities over students.

Before moving on, we first formally define the school choice problem and two widely used allocation methods, the Deferred Acceptance mechanism and the Boston mechanism.

4.1 The School Choice Problem

The *school choice problem* is one where a number of students are to be assigned to capacity constrained schools. Each student has a *strict* preference ordering over schools and each school has a *strict* priority ordering, possibly determined by a lottery, over students.

Formally, we have a set of students $\mathcal{I} = \{i_1, ..., i_m\}$, a set of schools $\mathcal{S} = \{s_1, ..., s_n\}$, where school $s \in \mathcal{S}$ has $q_s \geq 1$ available seats. Each student i has a strict preference profile

[21] Of course, the set of desirable objects may differ among agents.

$P_i = (P_i(1), ..., P_i(n))$, and each school s has a strict priority list $\pi_s = (\pi_s(1), ..., \pi_s(m))$ where for all j, k such that $1 \leq j < k \leq n$, student i prefers school $P_i(j)$ strictly more than school $P_i(k)$ and similarly, for each j, k such that $1 \leq j < k \leq m$, student $\pi_s(j)$ has a higher priority than student $\pi_s(k)$ at school s.

A matching is a function $\mu : \mathcal{I} \to \mathcal{S}$ such that $\# \{i \in \mathcal{I} : \mu(i) = s\} \leq q_s$ for all s, where $\mu(i) = s$ means that student i is assigned to school s.

A **school choice mechanism** is a procedure that constructs a matching for each school choice problem. We again implicitly consider school choice mechanisms to be direct mechanisms that ask students to report their preference profiles and implement the corresponding procedure given these reports and school priority lists.

We now describe two commonly used school choice mechanisms.

The Deferred Acceptance (DA) Mechanism

Step 1: Each student applies to his most preferred school. Each school *tentatively* accepts students one at a time following the priority order over students until there is no seat available or there is no other student applying to that school and rejects the remaining students, if any.

For $k \geq 2$,

Step k: Students who were rejected in Step $(k-1)$ apply to their next preferred school. Each school considers these students together *with* the students who were *tentatively* accepted in earlier steps and accepts students one at a time following the priority order over students until there is no seat available or there is no other student applying to that school and rejects the remaining students, if any.

The algorithm stops when there is no rejected student at some step. Students are assigned to the *final* school that they were tentatively accepted.

The Boston Mechanism

Students report a strict ranking over schools, and given these rankings and the school priorities over students, algorithm works as follows:

Step 1: Each student applies to his (reported) *first* choice. Each school accepts students one at a time following the priority order over students until there is no seat available or there is no other student applying to that school and rejects the remaining students, if any.

The number of available seats in each school is reduced by the number of students accepted by that school.

In general, for $k \geq 2$,

Step k: Students who were rejected in Step $(k-1)$ apply to their k^{th} choice. Seats of each school, if available, are assigned to those students one at a time following the priority order over students until there is no seat available at that school or there is no other student applying to that school and the remaining students, if any, are rejected. The number of available seats in each school is reduced by the number of students accepted by that school.

The algorithm stops when all seats of each school are filled or there are no unassigned students.

The main difference between these two procedures is that under the Boston mechanism each acceptance is final and immediate.[22]A student who ranks a school highly and gets accepted at earlier steps no longer faces any the competition from later applicants. On the other hand, each acceptance under the DA mechanism is tentative and the final matching is settled only after the last step. Hence, a student who ranks a school highly and is tentatively accepted still faces the competition with later applicants.

4.2 Welfare Comparison of School Choice Mechanisms

We again assume that agents' (students', here) preferences are uncorrelated. More precisely, each student's ranking over schools is independent of others' rankings and each possible strict ranking is (ex-ante) equally likely. Assume further that schools' preferences/priorities over students are identical[23] and each strict priority ranking over students is equally likely. Before moving on, recall that DA is strategy-proof but the Boston mechanism is not. However, when students' preferences and school priorities are as described, then truth-telling becomes a Bayes-Nash equilibrium when the number of available seats are identical for each school.[24] Therefore, we will compare these school choice mechanisms in terms of welfare of students under truth-telling.

[22] Thus, the Boston mechanism may be justly called the "Immediate Acceptance" mechanism as is sometimes referred in the literature.

[23] We refer the reader to Introduction for comments about this assumption.

[24] The proof is very similar to the proof that truth-telling is an equilibrium under the Restricted Ranking mechanism. Furthermore, Featherstone and Niederle (2013) also proves this for the Boston mechanism in a similar setting.

It is easy to observe that when each school has an identical ranking of students and each school has one available seat, then for any ranking over students the Deferred Acceptance mechanism yields the identical assignments with the Serial Dictatorship mechanism in which the order of agents (students, here) is according to this identical ranking of schools.[25] Therefore, when each possible ranking over students is equally likely, the interim allocation under DA is equivalent to the one under the Random Serial Dictatorship mechanism.[26] Second, we observe that the interim random allocation under the Boston mechanism is equivalent to the one under the Random Boston mechanism when each possible priority ranking over students is equally likely. This is not as immediate but it can be easily seen as follows. Under the Boston mechanism, a student competes only with students who applies to the same school at the same step. And, importantly, students applying to the same school at the same step are all alike because the only information regarding a student's rank in the priority list at step k is that all these students are eliminated in the earlier steps, that is, eliminated $(k-1)$ times, which only means that each student ranks below some $(k-1)$ students in the school priority lists and there is no additional information obtained. That is, any possible ranking over the students applying to a school at step k is still equally likely. Thus, at the interim stage, the computations are as if there is a random draw among students competing at step k as in the Random Boston mechanism.[27]

Hence, our main result that RB interim dominates RSD in a large market (Theorem 1) trivially implies the following result:

Proposition 2 *Assume that there is $M \geq 1$ such that the number of desirable schools is at most M. Assume further that each student's ranking over schools is independent of other students' rankings and each possible ranking is equally likely. In addition, each school's priority ranking over students is identical and each possible ranking is equally likely. When each school has one available seat, for sufficiently large n, the number of students and schools, the Boston mechanism strictly interim dominates the Deferred Acceptance mechanism.*

[25] See Balinski and Sönmez (1999), for example, for this equivalence.

[26] By interim allocation, we mean the random allocation that is computed before all the uncertainty are resolved (except that each agent knows his own preference). That is, under DA, the school priorities are still unknown to students and under RSD the random order is not realized yet.

[27] This is mostly due to the fact that different tie-breaking methods will yield the same (interim) random allocations under the Boston mechanism, which can be seen by similar discussions as above. The same, on the other hand, is not true for DA. See Abdulkadiroğlu, Pathak, and Roth (2009), for example.

5 Conclusion

This paper has studied allocation problems (the object allocation problem (one-sided preferences) and the school choice problem (two-sided preferences)) where $n \geq 2$ objects are to be allocated to $n \geq 2$ agents, and each agent is entitled to receive an object, in an incomplete information setting in which each agent's preference ranking is privately known. For the object allocation problem, we show that when each agent's preference ranking over objects is independently drawn from a uniform distribution, the celebrated RSD mechanism is dominated by another simple mechanism, the Random Boston mechanism, in terms of welfare of agents in a very strong manner. Regardless of the preference intensities of agents, each agent has a higher interim utility under the Random Boston mechanism in large markets. We also show that this result has an implication about the welfare comparison of two widely-used school choice mechanisms, the Boston mechanism and the Deferred Acceptance mechanism. Again in large markets the Boston mechanism is shown to be welfare superior to the Deferred Acceptance mechanism in the same strong manner when schools' priorities over students are identical and each school has one available seat.

Although symmetric and independent preferences as in our paper are commonly used in the literature in many mechanism design problems, it is admittedly restrictive. Still, it is important to understand how different allocation mechanisms perform in terms of welfare when agents' preferences may differ, which is the case in this paper. Furthermore, as discussed in the Introduction, our results would apply to cases in which there is grouping of objects/schools. That is, when, say, there are two groups of objects where everyone prefers the objects in the first group to the ones in the second group but agents' preferences within each group are uncorrelated. We should also note that since our results yield strict dominance, the conclusions will still hold in close enough settings. Nevertheless, it is of course desirable to obtain welfare comparison results when preferences are allowed to be more general. Furthermore, especially in the school choice context, it is limiting that each school has one available seat.[28] Maybe more importantly, in our setting the Boston mechanism is (Bayesian) incentive compatible, which eliminates the mostly criticized strategic disadvantage of the Boston mechanism. Regardless, together with the earlier results in the literature about the welfare comparison of these two school mechanisms, we now have two occasions, one with minimum diversity in preferences and one with maximum diversity in preferences,

[28] Although the computer simulations we run show that the same result holds when there are multiple copies (seats), we are unable to prove this analytically but we hope that ours will be a first theoretical step in this direction.

in which the Boston mechanism is welfare superior to the DA mechanism that has recently replaced the Boston mechanism in many school districts in the U.S. due to its better incentive properties. Although the discrete allocation problems studied here become easily intractable in more general environments in the case of incomplete information, we hope that techniques and results of this paper will be useful for further analysis.

A Appendix

A.1 Proof of Proposition 1 (Truth-telling under the Random Boston mechanism)

Assume that each agent's preference ranking over objects is independent of other agents' rankings and each possible ranking over objects is equally likely. We claim that truth-telling is an equilibrium under the Random Boston mechanism.

Let the set of agents be $\mathcal{I} = \{1, ..., m\}$, and the set of objects be $\mathcal{O} = \{o_1, ..., o_n\}$. Each agent i's preference profile be $R_i = (R_i(1), ..., R_i(n))$ where agent i prefers object $R_i(j)$ strictly more than object $R_i(k)$ for all j, k such that $1 \leq j < k \leq n$.

An (ordinal) allocation mechanism φ is an allocation rule $\varphi(R) = (\varphi_i(R))_i$ where $\varphi_i(R) = \left(\varphi_i^j(R)\right)_{j=1,\dots,n}$ such that for each $j \in \{1, ..., n\}$

$$\sum_{i=1}^{n} \varphi_i^j(R) = 1$$

and $\varphi_i^j(R) \in [0, 1]$ for all i and j where $\varphi_i^j(R)$ denotes the probability that agent i obtains object o_j when the reported ordinal preferences of agents are R.

We first present the following definition.

Definition 1 *An allocation mechanism is neutral iff for any R and for any permutation $\sigma : \mathcal{O} \to \mathcal{O}$ over objects we have that*

$$\varphi_i(R^\sigma) = (\varphi_i(R))^\sigma$$

where R^σ is the preference rankings of agents obtained from R according to permutation σ and $(\varphi_i(R))^\sigma$ is the allocation rule of agent i again obtained from $\varphi_i(R)$ according to permutation σ, that is, by changing the name of object o_j to $\sigma(o_j)$ for each j.

That is, an allocation mechanism is neutral iff the assignments do not depend on the label of objects. If we relabel the objects, assignments change accordingly.

We next note that when each ranking is equally likely, given that other agents report their true preferences, the interim probability of obtaining some object only depends on the ranking for that object under a neutral mechanism: Without loss of generality (W.l.o.g.), consider agent i_1 and consider two preference rankings

$$R_1^* = \left(R_1^*(1), ..., R_1^*(k-1), o_k, R_1^*(k+1), ..., R_1^*(n)\right)$$

and

$$R_1^{**} = \left(R_1^{**}(1), ..., R_1^{**}(k-1), o_k, R_1^{**}(k+1), ..., R_1^{**}(n)\right)$$

Note that although rankings for the other objects may be different in these preference rankings, k^{th} choice is object o_k in both. We claim that the interim probability of obtaining object o_k is the same for both rankings given that other agents report their true rankings. Let $P_k^{\varphi}(R_1)$ be the *interim* probability that agent i_1 obtains object o_k when he reports preference ranking is R_1 and other agents report their true rankings. Hence,

$$P_k^{\varphi}(P_1) = \sum_{R_{-1}=(R_2,...,R_m)} \varphi_1^k(R_1, R_{-1}) \Pr(R_{-1})$$

where summation is over all the possible rankings of other agents.

Now, consider the permutation σ over objects such that $\sigma(R_1^*(j)) = R_1^{**}(j)$ for all j. Note that, due to neutrality, since $\sigma(o_k) = o_k$, for any R_{-1} and π, we have that

$$\varphi_1^k(R_1^*, R_{-1}) = \varphi_1^k(R_1^{**}, (R_{-1})^{\sigma}) \tag{1}$$

where $(R_{-1})^{\sigma}$ is obtained from R_{-1} by relabeling objects according to permutation σ.

Now,

$$\begin{aligned} P_k^{\varphi}(R_1^{**}) &= \sum_{R_{-1}=(R_2,...,R_m)} \varphi_1^k(R_1^{**}, R_{-1}) \Pr(R_{-1}) \\ &= \sum_{R_{-1}=(R_2,...,R_m)} \varphi_1^k(R_1^{**}, (R_{-1})^{\sigma}) \Pr((R_{-1})^{\sigma}) \\ &= \sum_{R_{-1}=(R_2,...,R_m)} \varphi_1^k(R_1^*, R_{-1}) \Pr(R_{-1}) \\ &= P_k^{\varphi}(R_1^*) \end{aligned}$$

where third equality is due to (1) and the fact that $\Pr((R_{-1})^{\sigma}) = \Pr(R_{-1})$ since each possible ranking over the objects is equally likely and independent of others' rankings.

Hence, we will characterize a neutral allocation mechanism φ by $(P_k^{\varphi})_{k=1}^{n}$, where P_k^{φ} denotes the *interim* probability that an agent obtains his (reported) k^{th} ranked choice when all the remaining agents report their true rankings under the mechanism φ. Thus, truth-telling is an equilibrium under a neutral allocation mechanism iff P_k^{φ} is (weakly) decreasing in k.

Note that the Ranking mechanism is trivially neutral. We now claim that the interim probability that agent i_1 obtains his (reported) k^{th} ranked choice is decreasing in k when all the remaining agents report their true rankings under the Random Boston mechanism. But, this is almost immediate. Consider any state (any preference report of other agents and random draw when necessary) in which agent i_1 obtains some object, say o_1. Increasing the rank of that object will also guarantee that agent i_1 receives o_1. Furthermore, there may exist states such that agent i_1 does not obtain an object but by increasing the rank for that object agent i_1 may obtain that object at that state. Hence, given the observation that the interim probability of obtaining an object only depends on the rank for that object, the interim probability can only (weakly) increase once an object is given a higher ranking.

Hence, truth-telling is an equilibrium under the Random Boston mechanism.

A.2 Proof of Theorem 1

We prove Theorem 1 in several steps. We start with computing the interim probability of obtaining k^{th} choice under RSD.

Lemma 2 *When there are n objects and n agents, for each $k \in \{1, ..., n\}$, the interim probability of obtaining k^{th} ranked object under RSD, $(P_k^n)^{RSD}$, is:*

$$(P_k^n)^{RSD} = \frac{(n+1)}{k(k+1)n}$$

Proof. We will simply use P_k^n instead of $(P_k^n)^{RSD}$ when there is no danger of confusion. Let's compute P_k^n for each $k \geq 1$ and $n \geq k$.

Note that there is a recursive relation between two successive stages of RSD. At some step, some agent obtains an object, and no other agent will be able to obtain that object in later steps. Furthermore, the agent who obtained an object at that step will not get any other object in later steps. Hence, that agent and the object assigned to him is out of the problem at the beginning of the next step and we are in the isomorphic problem with one less agent and one less object. By using this observation, we will relate P_k^n to P_j^{n-1}, $j = 1, 2, ..., (n-1)$.

Consider an agent, say i_1, and w.l.o.g., say his preference profile is such that his k^{th} choice is object o_k for all $k \geq 1$. Let's start with $k = 1$ and consider the probability that agent i_1 obtains object o_1 when the number of agents (including i_1) and objects is n.

With probability $\frac{1}{n}$, i_1 will be the first in the randomly drawn order and, thus, will get his first choice, object o_1. With probability $\frac{n-1}{n}$ some other agent will be chosen in the first place. W.l.o.g., say agent i_2 has been chosen to be the first. If i_2's top choice is also o_1, which happens with probability $\frac{1}{n}$, i_1 can not obtain object o_1. With probability $\frac{n-1}{n}$, i_2's first choice is different from object o_1. In that case, i_2 will be assigned his favorite object, say o_2, and we will move to second step. Since one object, object o_2, and one agent, agent i_2, are removed in the first step, at the beginning of the second step, we have $(n-1)$ agents and $(n-1)$ objects and in that case the probability that i_1 obtains object o_1 is P_1^{n-1}. Hence,

$$P_1^n = \frac{1}{n} \times 1 + \frac{n-1}{n} \times \left(\frac{1}{n} \times 0 + \frac{n-1}{n} P_1^{n-1} \right)$$

or, simply

$$P_1^n = \frac{1}{n} + \frac{n-1}{n} \left(\frac{n-1}{n} P_1^{n-1} \right) \tag{2}$$

The claim of the Lemma for $k = 1$ is

$$P_1^n = \frac{n+1}{2n}$$

We will prove it by induction using (2). We now verify that if the claim of the Lemma is true for $(n-1)$, then the claim is true for n:

$$\begin{aligned} P_1^n &= \frac{1}{n} + \frac{n-1}{n} \left(\frac{n-1}{n} P_1^{n-1} \right) \\ &= \frac{1}{n} + \frac{n-1}{n} \left(\frac{n-1}{n} \frac{n}{2(n-1)} \right) \\ &= \frac{1}{n} + \frac{n-1}{2n} = \frac{n+1}{2n} \end{aligned}$$

which is what we wanted to show.

Note also that $P_1^1 = 1$. (If there is only one agent, he will get his first choice for sure.) That is, the claim is also true for $n = 1$ and hence by induction hypothesis, we have the result.

Similarly, consider the probability of his getting his k^{th} choice, $n \geq k \geq 2$, when there are n agents and n objects.

With probability $\frac{1}{n}$, agent i_1 will be chosen in the first place and will get his first choice in this case, which means that he won't get his k^{th} choice object for $k \geq 2$. With probability $\frac{n-1}{n}$, some other agent, say agent i_2, will be chosen in the first place. If i_2's first choice is one of i_1's top $(k-1)^{th}$ choice, one of $o_1, .., o_{(k-1)}$, which happens with probability $\frac{k-1}{n}$, object o_k becomes i_1's $(k-1)^{th}$ choice among the remaining $(n-1)$ objects. And, with probability $\frac{1}{n}$, i_2's first choice is object o_k. In this case, i_2 will receive o_k and hence i_1 won't be able to get o_k. With probability $\frac{n-k}{n}$, agent i_2's top choice is different from one of i_1's top k choices. That is, his top choice is from the set $\{o_{k+1}, ..., o_n\}$. In that case object o_k remains to be agent i_1's k^{th} choice among the remaining $(n-1)$ objects. Therefore, for $n \geq k \geq 2$,

$$P_k^n = \frac{n-1}{n}\left[\frac{k-1}{n}P_{k-1}^{n-1} + \frac{n-k}{n}P_k^{n-1}\right]$$

It is easy to verify that for any $k \geq 2$, if the claim of the lemma is true for $(n-1)$, it is true for n :

$$\begin{aligned}
P_k^n &= \frac{n-1}{n}\left[\frac{k-1}{n}P_{k-1}^{n-1} + \frac{n-k}{n}P_k^{n-1}\right] \\
&= \frac{n-1}{n}\left[\frac{k-1}{n}\frac{n}{k(k-1)(n-1)} + \frac{n-k}{n}\frac{n}{k(k+1)(n-1)}\right] \\
&= \frac{n-1}{n}\left[\frac{1}{k(n-1)} + \frac{n-k}{k(k+1)(n-1)}\right] \\
&= \frac{n-1}{n}\left[\frac{n+1}{k(k+1)(n-1)}\right] \\
&= \frac{(n+1)}{k(k+1)n}
\end{aligned}$$

Note that $P_2^2 = 1 - P_1^2 = \frac{1}{4}$ since $P_1^n = \frac{n+1}{2n}$ by above and hence by induction we have that $P_2^n = \frac{n+1}{6n}$. Given this, $P_3^3 = 1 - P_1^3 - P_2^3 = 1 - \frac{2}{3} - \frac{2}{9} = \frac{1}{9}$ and again by induction, we have that $P_3^n = \frac{n+1}{12n}$. Continuing in this manner, for a general $k \geq 2$, we have

$$\begin{aligned}
P_k^k &= 1 - \sum_{j=1}^{k-1} P_j^k \\
&= 1 - \sum_{j=1}^{k-1} \frac{k+1}{j\,(j+1)\,k} \\
&= 1 - \frac{k+1}{k} \sum_{j=1}^{k-1} \left(\frac{1}{j} - \frac{1}{j+1} \right) \\
&= 1 - \frac{k+1}{k} \left(\frac{k-1}{k} \right) = \frac{1}{k^2}
\end{aligned}$$

and hence by induction we have that

$$P_k^n = \frac{(n+1)}{k\,(k+1)\,n}$$

which is the desired result. ■

Although we are able to explicitly calculate these probabilities for the RSD, unfortunately, it is not easy to calculate the probabilities for each n and k under the Random Boston mechanism. One tractable case is when $k = 1$. That is, the probability of obtaining top choice. Assume that there n agents and n objects. Consider an agent, say i_1, and let's compute his probability of getting his first choice, again w.l.o.g. say his first choice is object o_1. There are $(n-1)$ remaining agents. Denote the number of agents (other than agent i_1) whose first choice is object o_1 be $j \in \{0, ..., n-1\}$. The probability of agent i_1 getting object o_1 is $\frac{1}{j+1}$ when there are j other agents whose first choice is o_1. Hence,[29]

$$
\begin{aligned}
P_1^n &= \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-1-j} \left(\frac{1}{j+1}\right) \\
&= \sum_{j=0}^{n-1} \binom{n}{j+1} \left(\frac{1}{n}\right)^{j+1} \left(\frac{n-1}{n}\right)^{n-1-j} \\
&= \sum_{j=1}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-j} \\
&= \left[\sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-j}\right] - \left(\frac{n-1}{n}\right)^n \\
&= 1 - \left(\frac{n-1}{n}\right)^n
\end{aligned}
$$

As a future reference note that $P_1^n \to 1 - \frac{1}{e}$ as $n \to \infty$. For $k > 1$, by computing average number of objects assigned at each step, we compute the limit probabilities as $n \to \infty$. Before moving on, we make couple of observations regarding the computation.

Under the Random Boston mechanism if an agent, say agent i_1, obtains some object in step k, this means that agent i_1 obtains his k^{th} choice. Furthermore, the number of unassigned objects after the allocation has been made at Step k is equivalent to the number of agents that should receive an object in later steps.

Note also that under the Random Boston mechanism, for any object, an agent competes only with agents who rank the object at the exact same rank. Thus, the probability that an agent receives an object at some step when there are, say j, other agents rank that object same and haven't received an object at earlier steps is $\frac{q}{j+1}$ when $j \geq q$ and 1 if $j < q$ where q is the number of objects unassigned in earlier steps.

These observations will be key in the computation below.

Note finally that the probability of k "successes" out of N independent "trial"s when the probability of success is p is given by the Binomial Distribution

$$b\,(k; N, p) = \binom{N}{k} p^k \,(1-p)^{N-k}$$

and in particular,

$$b\,(0; N, p) = (1-p)^N$$

[29] Again, when there is no danger of confusion, we simply write $(P_k^n)^{RR}$ as P_k^n.

Assume that there are n agents and n objects. We want to compute the interim probability of obtaining k^{th} choice under the Random Boston mechanism as $n \to \infty$. We will do so by computing the (average) number of agents obtaining an object at step k.

Step 1: Now, for any object, the probability that no agent ranks it as a first choice is

$$b\left(0; n, \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n$$

Note that

$$r_1^n = \left(\frac{n-1}{n}\right)^n \to \frac{1}{e}$$

as $n \to \infty$. Call

$$q_1 = \frac{1}{e}$$

Hence, for any object, the probability that at least one agent ranks it as a first choice and hence an agent obtains an object is $1 - r_1^n$. Therefore, in average there are nr_1^n objects unassigned in Step 1. In other words, $n(1 - r_1^n)$ agents gets their top choice. Hence, by symmetry and there are n agents,

$$P_1^{RB} \to 1 - \frac{1}{e}$$

as $n \to \infty$.

Step 2: Now, in average there are nr_1^n agents who haven't obtained an object in Step 1 and hence nr_1^n agents to be assigned in Step 2. For any object, the probability that there is no agent (among the ones who haven't been assigned an object in Step 1) who ranks this object as a second choice

$$b\left(0; nr_1^n, \frac{1}{n-1}\right) = \left(\frac{n-2}{n-1}\right)^{nr_1^n}$$

since there are nr_1^n agents and there are $(n-1)$ objects that an agent can rank as a second choice.[30]

Let

$$r_2^n = \left(\frac{n-2}{n-1}\right)^{nr_1^n}$$

[30] Note that nr_1^n may not be an integer. We are making an abuse of notation here but we are interested in the limit probabilities.

and

$$q_2 = \lim_{n\to\infty} \left(\frac{n-2}{n-1}\right)^{nr_1^n} = \exp\left(-q_1\right)$$

Thus, there remains (in average) $nr_1^n r_2^n$ objects that haven't been assigned. In other words, $nr_1^n\left(1-r_2^n\right)$ agents obtained an object in step 2. Hence, as $n\to\infty$

$$P_2^{RB} \to q_1\left(1-q_2\right)$$

In general,

Step k: There are in average $nr_1^n...r_{k-1}^n$ active agents. For any object, the probability that no agent ranks it as k^{th} choice is

$$b\left(0; nr_1^n...r_{k-1}^n, \frac{1}{n-(k-1)}\right) = \left(\frac{n-k}{n-k+1}\right)^{nr_1^n...r_{k-1}^n}$$

and $n\left(\prod_{s=1}^{k-1} r_s^n\right)\left(1-r_k^n\right)$ objects are assigned at step k where

$$r_k^n = \left(\frac{n-k}{n-k+1}\right)^{nr_1^n...r_{k-1}^n}$$

Note that

$$q_k = \lim_{n\to\infty} \left(\frac{n-k}{n-k+1}\right)^{nr_1^n...r_{k-1}^n} = \exp\left(-q_1...q_{k-1}\right)$$

hence, as $n\to\infty$,

$$P_k^{RB} \to \left(\prod_{j=1}^{k-1} q_j\right)\left(1-q_k\right)$$

Just to see what these numbers look like, the table below shows the limit probabilities under RB for $1\le k\le 10$ and the probabilities obtained by computer simulations for $n=25$.[31]

[31] Simulations have been done as follows. We fix an agent, say agent i_1, and randomly draw preference rankings for other agents and check what object i_1 receives. To get the numbers listed, we repeated this for $m = 500000$ times and checked how many times i_1 gets his first choice, second choice and so on.

Furthermore, we also list the interim probabilities for RSD for $n = 25$.

k	$P_k^{RB}(n \to \infty)$	$(P_k^{25})^{RB}$	$(P_k^{25})^{RSD}$
1	0.632121	0.6389	0.52
2	0.113233	0.1172	0.160256
3	0.057247	0.0600	0.08667
4	0.035362	0.0371	0.052
5	0.024239	0.0258	0.032051
6	0.017738	0.0190	0.024762
7	0.013583	0.0144	0.018571
8	0.010755	0.0117	0.013355
9	0.008738	0.0096	0.011556
10	0.007247	0.008	0.009455

We now turn back and prove Lemma 1, that is, we want to show

$$\lim_{n\to\infty} \sum_{k=1}^{K} P_k^B > \lim_{n\to\infty} \sum_{k=1}^{K} P_k^{DA}$$

for all $K \geq 1$.

To do so, let's consider the limit probabilities. ($(P_k^n)^{RB}$ and $(P_k^n)^{RSD}$ for each k as $n \to \infty$) We will make an abuse of notation and denote the limit probabilities P_k^{RB} and P_k^{RSD} when there is no confusion. First, by Lemma 2 we have that for all $k \geq 1$

$$(P_k^n)^{RSD} = \frac{n+1}{k\,(k+1)\,n}$$

Thus, in the limit,

$$P_k^{RSD} = \frac{1}{k\,(k+1)}$$

Recall that

$$P_k^{RB} = \left(\prod_{j=1}^{k-1} q_j\right)(1 - q_k)$$

where

$$q_1 = \frac{1}{e}$$

and

$$q_k = \exp\left(-\prod_{s=1}^{k-1} q_s\right)$$

for all $k \geq 2$.

Given these, we prove Lemma 1. That is, we want to show that for all $K \geq 1$

$$\lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^{RB} > \lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^{RSD}$$

The proof goes as follows:

Now

$$P_k^{RB} = \left(\prod_{j=1}^{k-1} q_j\right)(1-q_k)$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{K} P_k^{RB} &= \sum_{k=1}^{K} q_1 \ldots q_{k-1}(1-q_k) \\
&= (1-q_1) + q_1(1-q_2) + q_1 q_2 (1-q_3) + \ldots + q_1 \ldots q_{K-1}(1-q_K) \\
&= 1 - q_1 q_2 \ldots q_{K-1} q_K
\end{aligned}$$

and we know that

$$(P_k^n)^{RSD} = \frac{n+1}{k(k+1)n}$$

Hence, the limit probability is

$$P_k^{RSD} = \frac{1}{k(k+1)}$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{K} P_k^{RSD} &= \sum_{k=1}^{K} \frac{1}{k(k+1)} \\
&= \sum_{k=1}^{K} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 1 - \frac{1}{K+1}
\end{aligned}$$

We, now, claim that

$$1 - q_1 q_2 ... q_{K-1} q_K > 1 - \frac{1}{K+1}$$

That is, we want to show that

$$q_1 q_2 ... q_{K-1} q_K < \frac{1}{K+1}$$

Let us define

$$r_K = \ln q_K$$

for each $K \geq 1$. Now,

$$r_{K+1} = \ln q_{K+1} = \ln \left(\exp \left(-q_1 ... q_K \right) \right) = -q_1 ... q_K$$

and

$$r_K = \ln q_K = \ln \left(\exp \left(-q_1 ... q_{K-1} \right) \right) = -q_1 ... q_{K-1}$$

$\Longrightarrow$

$$\begin{aligned} r_{K+1} &= r_K q_K \\ &= r_K e^{r_K} \end{aligned}$$

By definition

$$q_1 q_2 ... q_{K-1} q_K = -\ln \left(q_{K+1} \right)$$

Thus, we want to show that for all $K \geq 1$

$$-\ln \left(q_{K+1} \right) < \frac{1}{K+1}$$

or for all $K \geq 2$

$$r_K > -\frac{1}{K}$$

We prove this by induction. The claim is true for $K = 2$:

$$r_1 = \ln q_1 = -1$$

$$r_2 = -e^{-1} = -\frac{1}{e} > -\frac{1}{2}$$

Assume that the claim is true for $K = n$, we want to verify the relation for $n+1$. Note that

$$r_{n+1} = f(r_n)$$

where

$$f(x) = xe^x$$

Note that

$$f'(x) = e^x + xe^x \geq 0$$

for $x \geq -1$. Now,

$$r_{n+1} = f\left(-\frac{1}{n}\right) = -\frac{1}{n}e^{-\frac{1}{n}} \tag{3}$$

We want to show that

$$r_{n+1} > -\frac{1}{n+1}$$

We claim that

$$e^{-\frac{1}{n}} < \frac{n}{n+1}$$

or equivalently

$$e^{\frac{1}{n}} > \frac{n+1}{n} = 1 + \frac{1}{n}$$

Define

$$h(x) = e^x - x - 1$$

Note that

$$h'(x) = e^x - 1 > 0 \text{ for } x > 0.$$

Furthermore,

$$h(0) = 0$$

Hence, for all $n > 1$,

$$h\left(\frac{1}{n}\right) > 0$$

Then, it is easy to verify

$$e^{-\frac{1}{n}} < \frac{n}{n+1}$$

Therefore,

$$r_{n+1} = f(r_n) \geq f\left(-\frac{1}{n}\right) = -\frac{1}{n}e^{-\frac{1}{n}} > -\frac{1}{n+1}$$

where the weak inequality comes from the fact that f is increasing together with the induction hypothesis and strict inequality from (3). Hence, we have

$$\sum_{k=1}^{K} P_k^{RB} > \sum_{k=1}^{K} P_k^{RSD}$$

which is the desired result.

Now, we are ready to prove Theorem 1:

Given M, the number of desirable objects, we know that we can find large enough N such that for $n > N$

$$\sum_{k=1}^{K} (P_k^n)^{RB} > \sum_{k=1}^{K} (P_k^n)^{RSD}$$

for all $K \leq M$.

Take some $n > N$. Take any agent and assume without loss of generality that his preference is such that he derives a utility v_k^n from object o_k with $v_j^n > v_k^n > 0$ for each $j < k < M$ and $v_k^n = 0$ for $k > M$. We want to show that

$$\sum_{k=1}^{n} (P_k^n)^{RB} v_k^n > \sum_{k=1}^{n} (P_k^n)^{RSD} v_k^n$$

Now,

$$\begin{aligned}
& \sum_{k=1}^{n} (P_k^n)^{RB} v_k^n \\
= \;& (P_1^n)^{RB} (v_1^n - v_2^n) + \left((P_1^n)^{RB} + (P_2^n)^{RB}\right) (v_2^n - v_3^n) + ... + \\
& \left((P_1^n)^{RB} + ... + \left(P_{n-1}^n\right)^{RB}\right) \left(v_{n-1}^n - v_n^n\right) + \left((P_1^n)^{RB} + ... + (P_n^n)^{RB}\right) v_n^n \\
= \;& v_n^n + \sum_{k=1}^{n-1} \left((P_1^n)^{RB} + ... + (P_k^n)^{RB}\right) \left(v_k^n - v_{k+1}^n\right)
\end{aligned}$$

since $(P_1^n)^{RB} + ... + (P_n^n)^{RB} = 1$. Similarly,

$$\begin{aligned}
& \sum_{k=1}^{n} (P_k^n)^{RSD} v_k^n \\
= \;& v_n^n + \sum_{k=1}^{n-1} \left((P_1^n)^{RSD} + ... + (P_k^n)^{RSD}\right) \left(v_k^n - v_{k+1}^n\right)
\end{aligned}$$

Thus,

$$
\begin{aligned}
& \sum_{k=1}^{n}\left((P_k^n)^{RB}-(P_k^n)^{RSD}\right)v_k^n \\
= & \sum_{k=1}^{n-1}\left[\left((P_1^n)^{RB}+...+(P_k^n)^{RB}\right)-\left((P_1^n)^{RSD}+...+(P_k^n)^{RSD}\right)\right]\left(v_k^n-v_{k+1}^n\right) \\
= & \sum_{k=1}^{M}\left[\left((P_1^n)^{RB}+...+(P_k^n)^{RB}\right)-\left((P_1^n)^{RSD}+...+(P_k^n)^{RSD}\right)\right]\left(v_k^n-v_{k+1}^n\right) \\
> & \ 0
\end{aligned}
$$

since for $n > N$, $\left((P_1^n)^{RB}+...+(P_k^n)^{RB}\right) > \left((P_1^n)^{RSD}+...+(P_k^n)^{RSD}\right)$ and $v_k^n > v_{k+1}^n$ for all $k \leq M$, and $v_k^n = 0$ for all $k > M$. This is what we wanted to show.

Acknowledgements. I am deeply indebted to Vijay Krishna for his support and for numerous comments and suggestions. I also thank seminar audience at Carnegie Mellon University, Penn State University, Maastricht University, University of Technology Sydney, METU, New Economic School, and TOBB-ETU for their questions and comments and several participants of Midwest Economic Theory and Stony Brook Game Theory conferences for discussions about the project at earlier stages. Needless to say, all the remaining errors and/or oversights are mine. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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