Allocation with Points: Colonel Blotto Game under Incomplete Information

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Abstract

We consider the problem of allocating multiple objects to agents via an auction by using "points" as in the "Course Bidding System" which is employed by several business and law schools to allocate course seats to students. We assume that each agent has a fixed amount of divisible "points" which can only be used for bidding purposes and have no monetary value. Each agent simultaneously bids for the objects, and each object is allocated to the agent who bids highest for that object. This game is equivalent to the classical *Colonel Blotto* game. We consider an incomplete information setting where each agent has *multidimensional private information* regarding valuations for multiple objects and solve for a Bayes-Nash equilibrium of this game for a class of value distributions when there are two agents and $n \geq 2$ objects. Furthermore, we show that for all the value distributions for which we can solve for equilibrium in closed form, every type of agent has a higher interim payoff under this allocation method compare to any other allocation method that depends only on ordinal preferences.

Keywords: Colonel Blotto game; incomplete information; allocation without transfers.

JEL Codes: C7; D82

1 Introduction

It is unfair or undesirable to use monetary transfers in many real-life situations. Allocating office spaces to workers, students to public schools, course seats to students are notable examples among many others. It is also important to increase efficiency to the maximum possible extent in such situations. Recently, many business and law schools have adopted a system based on an auction where "points" are used for bidding to allocate course seats to students, the so-called "Course Bidding System", in order to create a fake market environment in the absence of monetary transfers in the hope of allowing students with the highest desire for a course to take that particular course.

Although there are several complexities, this system mainly allocates course seats to students as follows. Students are given a positive bid endowment, the so-called points,

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and they bid for the courses by using these points. Students with highest bids for a course, up to the number of available seats, are assigned a seat in that course.[1]

Inspired by this system, we consider the problem of allocating goods to agents via an auction where only "points" can be used for bidding.[2] In particular, we assume that there are $n \geq 2$ distinct objects to be allocated to $m = 2$ agents. We consider a standard Bayesian setting in which each agent has private information regarding his/her values for the objects. Each agent is endowed with divisible points in the amount of $B > 0$ which is commonly known and can only be used for bidding. Each agent simultaneously submits bids for each object under the "budget" constraint. Then, each object is given to the agent who submits the highest bid.[3] We, furthermore, assume that preferences of agents over bundles of objects are additively separable. That is, the payoff of an agent from a bundle is just the sum of his/her values for the objects included in that bundle.

The auction game considered here is equivalent to the classical Colonel Blotto game, which was introduced by Borel (1921). In the original version of this game, there are two colonels and each has a unit of military resources. The colonels are going to fight in several battlefields.[4] Each colonel simultaneously chooses how to distribute limited resources across the battlefields. At each battlefield, the colonel who has higher resources wins and each colonel's payoff is equal to the number of battlefields at which he has won the fight. That is, each battlefield is equally valued by each colonel. Note also that colonels only care whether they win the fight at a battlefield. That is, gone resources are sunk costs that do not affect colonels' payoffs.

Although the Colonel Blotto game has been first considered in such a military situation, it has several applications in a variety of political, social and other competitive situations in addition to the course bidding system as mentioned. Political campaigns constitute one example in which a similar situation arises. Political parties should decide how to allocate their resources to attract voters, and they only care whether they win the election. Another application might be trials in which lawyers have to decide how much resource (time, effort) to allocate into different lines of defense to be able to "win" against other lawyer. Several other situations such as lobbying, competing in sports can be considered as applications of this game.

We consider the Colonel Blotto game when each agent's values for objects are privately known and agents' values may be different from other agents' values. More precisely, each agent's private information is an n-dimensional vector that consists of values for each object. Our results are two-fold. First, we look for a Bayes-Nash equilibrium of this game. Although it is in general hard to obtain closed-form expressions for equilibria when there is multi-dimensional incomplete information in such problems, we were able to do so in some cases. We first consider the case when there are $n = 2$ objects. In that case, agents have a (weakly) dominant strategy to bid all their points in the object that

[1] Northwestern Kellogg, MIT Sloan, Wharton, Yale School of Management, Columbia Business School and University of Michigan Business School are among many others that employ variants of such an auction to allocate course seats to students. We refer the reader to Sönmez and Ünver (2010) for further details about this system.

[2] Note that we will not try to formalize and/or solve for the "Course Bidding System". We will consider a simpler allocation problem. We just borrow the idea of "allocation with points" from this system.

[3] Ties are resolved by a fair coin toss.

[4] In the original version, there are 3 battlefields.

they value more. Note that this strategy does not depend on any information regarding intensity of the preferences. It just depends on the ordinal ranking over objects. Next, we consider the case when $n > 2$. In this case, the equilibrium strategies depend on the intensity of preferences, namely on cardinal preferences. For a class of value distributions, we obtain simple closed-form expressions for an equilibrium in this case.

The original Colonel Blotto game dates back to 1921. The solution to this game is given by Borel and Ville (1938) when there are 3 battlefields, and later by Gross and Wagner (1953) for more than 3 battlefields, and valuations are identical both across battlefields and agents. Roberson (2006) analyzes the game allowing for asymmetric budgets. Hortala-Vallve and Llorente-Saguer (2012) and Thomas (2018) consider the problem when colonels have asymmetric and heterogeneous battlefield valuations. Importantly, all these papers consider the problem under complete information: Each colonel knows the battleship valuations of the other colonel and also know the budget (total resources) of the other colonel.

There are just a couple of papers that investigate the Colonel Blotto game under *incomplete* information. Adamo and Matros (2009) study the Colonel Blotto game when there is incomplete information about agents' *budgets* but battleship valuations are *commonly known*. Kovenock and Roberson (2011) (KR, henceforth) consider an environment similar to ours in which agents have private information about their valuations and agents' budgets are symmetric and commonly known. KR solve for equilibrium for the case of 3 battlefields and 2 agents when each agent's value is independently and symmetrically drawn from one special value distribution, namely, the uniform distribution over the unit sphere on the nonnegative orthant for which the marginal distributions are uniform distribution over $[0, 1]$. In a more recent paper, Ewerhart and Kovenock (2019) extend the example of KR by solving for equilibrium for the case of $(n + 1)$ battlefields and n agents when each agent's value is again independently and symmetrically drawn from the uniform distribution over the generalized unit sphere for which the marginal distributions are still uniform distribution over $[0, 1]$ as in KR. We, on the other hand, solve for equilibrium for the case of $n \geq 2$ battlefields and two agents when agents' values are drawn (possibly) *asymmetrically* for a larger class of value distributions, not restricted to the uniform distribution, and the values are allowed to lie in more general domains than the unit sphere.

Second aspect of our results will be on welfare properties of this allocation method, which are nonexistent in related papers. In many real-life applications, allocation rules that only use ordinal preferences of agents over agents are predominantly used when transfers can not be used. In assigning students to public schools, students report their *ordinal* rankings over schools and a predetermined algorithm determines the assignments given students' ordinal preferences. The so-called Boston mechanism and the well-known Deferred Acceptance (DA) mechanism are currently the most widely-used mechanisms in many school districts in the US.[5] Another ordinal mechanism, the Random Serial Dictatorship (RSD) mechanism in which an order over agents is randomly determined and following the order each agent chooses the object he wants among the available

[5] Although school choice problem is different from the problem at hand since in school choice problem there are multiple copies of objects (school seats) and each agent (student) can only obtain one object and also it is a two-sided problem, we present these examples to emphasize the extensive use of ordinal mechanisms in practice.

objects, is a very popular mechanism that has been used extensively in many allocation problems when monetary transfers are not allowed.

Although assignment rules that only use ordinal preferences are predominantly used in applications and in the literature, it is not clear why we should restrict ourselves to such rules.[6] "Auction with points"[7] is another way to allocate objects to agents without monetary transfers, as in allocating course seats to students, and its outcomes can depend on not only ordinal preferences but also cardinal preferences.[8] Therefore, it is important to understand how this mechanism compares to the other mechanisms in terms of welfare of agents. For the cases we could obtain closed-form expressions for equilibrium for the Blotto game, we compare allocating objects via an auction with points to incentive compatible ordinal mechanisms and we show that *each type* of agent has a higher interim payoff under the Blotto game than any other ordinal mechanism. Allocating via an auction with points is a mechanism that is easy to implement in many real-life situations, which is already used in practice as in the aforementioned Course Bidding System. Therefore, welfare superiority of this mechanism over the widely-used ordinal mechanisms may have significant policy implications.

The remainder of this paper proceeds as follows. First, we introduce the formal model, and we solve for equilibrium first for $n = 2$ and then we obtain equilibrium for a class of value distributions when $n \geq 3$. In Section 3, we compare the welfare of agents under the Blotto mechanism to ordinal mechanisms. Finally, we conclude.

2 Model

Assume that there are $m = 2$ agents and $n \geq 2$ objects. Each agent i derives a value $v_j^i \geq 0$ from obtaining object j. Assume that $\mathbf{v}^i = (v_1^i, ..., v_n^i)$ is independently drawn from a distribution G_i over $\left[\underline{v}^i, \overline{v}^i\right]^n$. The payoff of an agent is just the sum of his values of objects he gets.

Each agent has divisible bid endowment (points) in the amount of $B > 0$ and this is commonly known. Agents, after privately observing their valuations, simultaneously choose how much to bid for each object where the sum of bids is B.[9] That is, agent i chooses a bid vector $(b_1^i, ..., b_n^i) \in [0, B]^n$ such that

$$\sum_{j=1}^{n} b_j^i = B.$$

For each object, the agent who has the highest bid for that object gets the object and if there is a tie, the agent who gets the object is determined by a fair coin toss. Monetary transfers are not allowed and only points can be used for bidding.

We want to solve for Bayesian Nash equilibrium of this game. We denote a pure strategy as $\beta^i : \left[\underline{v}^i, \overline{v}^i\right]^n \to [0, B]^n$ such that $\beta^i = \left(\beta_j^i\right)_{j=1}^n$ and for all $\mathbf{v}^i = (v_1^i, ..., v_n^i)$,

[6] Che and Kojima (2010) similarly states that only using ordinal preferences in many assignment rules is unclear but they take it as given in Footnote 3.

[7] We will sometimes refer to this method as the "Blotto mechanism" from now on.

[8] Except the case when there are $n = 2$ objects as we have discussed above.

[9] Since points have no value other than bidding, agents will always use all of their points.

$\beta_j^i\left(\mathbf{v}^i\right) \in [0, B]$ and

$$\sum_{j=1}^{n} \beta_j^i\left(\mathbf{v}^i\right) = B.$$

Firstly, we consider the case when $n = 2$.

Proposition 1 *Assume that there are $n = 2$ objects to be allocated and agent i's values are independently drawn from distribution $G_i(.,.)$. Then, a strategy in which each agent bids all points on the object with a higher valuation is a weakly dominant strategy. More precisely, any strategy with*

$$\beta(v_1, v_2) = \begin{cases} (B, 0) & \text{if} \quad v_1 > v_2 \\ (0, B) & \text{if} \quad v_1 < v_2 \end{cases}$$

is a weakly dominant strategy for each agent. That is,

Proof. Consider agent 1 with type (v_1, v_2). Assume that $v_1 > v_2$. Consider a strategy of agent 2 and let B_1^2 be the random variable denoting agent 2's bids on o_1. Now, by bidding $(B, 0)$, the expected payoff of agent 1 is

$$\Pr\left(B_1^2 = B\right)\left(\frac{v_1 + v_2}{2}\right) + \Pr\left(B_1^2 < B\right) v_1 \tag{1}$$

since if agent 2 bids B on first object this means that bids of agents for both objects are the same, therefore, each agent will get each object with probability $\frac{1}{2}$ and when if agent 2 bids less than B on o_1 which means that by bidding a positive amount on o_2, agent 1 will get o_1 and agent 2 will get o_2.
Similarly, by bidding $(b, 1-b)$, $0 < b < 1$, agent 1's expected payoff is

$$\begin{aligned} & \Pr\left(B_1^2 > b\right) v_2 + \Pr\left(B_1^2 = b\right)\left(\frac{v_1 + v_2}{2}\right) + \Pr\left(B_1^2 < b\right) v_1 \qquad (2) \\ = \; & \Pr\left(B_1^2 = B\right) v_2 + \Pr\left(B > B_1^2 > b\right) v_2 + \Pr\left(B_1^2 = b\right)\left(\frac{v_1 + v_2}{2}\right) + \Pr\left(B_1^2 < b\right) v_1. \end{aligned}$$

By bidding $(0, B)$, agent 1's expected payoff

$$\Pr\left(B_1^2 > 0\right) v_2 + \Pr\left(B_1^2 = 0\right)\left(\frac{v_1 + v_2}{2}\right). \tag{3}$$

Note that $(1) \geq (2) \geq (3)$. Thus, when $v_1 > v_2$, for any bid distribution of agent 2, B_1^2, it is (weakly) better to bid $(B, 0)$ than any other possible bid. Similarly, for $v_1 < v_2$, for any bid distribution of agent 2, B_1^2, it is (weakly) better to bid $(0, B)$ than any other possible bid.. For $v_1 = v_2$, agent is indifferent among all possible bids.

Furthermore, note that the above arguments similarly applies to agent 2. Hence, we have the desired result. ■

Note that when there are $n = 2$ objects, the equilibrium of this game only depends on *ordinal* preferences. More precisely, an individual bids all his points on the good he prefers more regardless of the intensity of preferences. However, when there are $n > 2$

objects, this is not true anymore and equilibrium strategies can depend on the preference intensities for goods as the results below will demonstrate.

First, we solve for equilibrium of this game for a certain class of value distributions when $n = 3$.

Proposition 2 *Assume that agent* 1 *and* 2*'s values are independently drawn from continuous distributions* $G_1(v_1, v_2, v_3)$ *and* $G_2(w_1, w_2, w_3)$*, respectively, such that densities are of the following form:*

$$g_1(v_1, v_2, v_3) = \widetilde{g}_1\left(\left(v_1^2 + v_2^2 + v_3^2\right)\right)$$

and

$$g_2(w_1, w_2, w_3) = \widetilde{g}_2\left(\left(w_1^2 + w_2^2 + w_3^2\right)\right),$$

where $\widetilde{g}_1$ *and* $\widetilde{g}_2$ *are measurable functions on* $\mathbb{R}_+$ *such that*

$$\int_0^\infty \widetilde{g}_i(x)\, x^{\frac{1}{2}} dx = \frac{4}{\pi}.$$ [10]

Then, the following is a symmetric equilibrium

$$\beta(v_1, v_2, v_3) = \left(\frac{v_1^2}{v_1^2 + v_2^2 + v_3^2}B, \frac{v_2^2}{v_1^2 + v_2^2 + v_3^2}B, \frac{v_3^2}{v_1^2 + v_2^2 + v_3^2}B\right).$$

Proof. See the Appendix. ■

That is, as long as the density functions depend only on the sum of squares, regardless of the support of the distribution and without requiring symmetry across agents, we obtain an equilibrium in closed-form. Furthermore, for any measurable function $\widetilde{g}_i$ that satisfies the given requirement, g_i becomes a density.

Some examples of densities that satisfy the given condition are as follows.

Example 1 *Let* $\widetilde{g}(x) = \frac{8}{\pi\sqrt{\pi}} \exp(-(x))$ *when* $x \geq 0$*. Note that*

$$\int_0^\infty \frac{8}{\pi\sqrt{\pi}} \exp(-(x))\, x^{\frac{1}{2}} dx = \frac{4}{\pi}.$$

Hence, for $g(v_1, v_2, v_3) = \frac{8}{\pi\sqrt{\pi}} \exp\left(-\left(v_1^2 + v_2^2 + v_3^2\right)\right)$*,* $v_1, v_2, v_3 \geq 0$ *we have the result. Note that this is the density if* v_1, v_2, v_3 *comes independently from Generalized Gamma Distribution with parameters* $(a, d, p) = (1, 1, 2)$ *where general density of this distribution is given by* $\left(\frac{p}{a^d} \frac{x^{d-1} e^{-\left(\frac{x}{a}\right)^p}}{\Gamma\left(\frac{d}{p}\right)}\right)$*,* $x \geq 0$*. That is, each* v_i *is independently from a distribution with density*

$$f(x) = \frac{2}{\sqrt{\pi}} \exp(-x),\ x \geq 0.$$

[10] This makes sure that $g_i(v_1, v_2, v_3)$ is a density function.

Example 2 *Let* $\widetilde{g}(x) = \frac{6}{\pi}$ *if* $0 \leq x \leq 1$ *and* 0 *otherwise.Then,*

$$g(v_1, v_2, v_3) = \frac{6}{\pi} \text{ if } v_1^2 + v_2^2 + v_3^2 \leq 1.$$

That is, if (v_1, v_2, v_3) *is uniformly distributed inside the unit sphere.*

Example 3 *Let* $\widetilde{g}(x) = \frac{6}{7\pi}$ *if* $1 < x \leq 4$ *and* 0 *otherwise.Then,*

$$g(v_1, v_2, v_3) = \frac{6}{7\pi} \text{ if } 1 < v_1^2 + v_2^2 + v_3^2 \leq 4.$$

That is, if (v_1, v_2, v_3) *is uniformly distributed inside sphere with radius* 2 *but outside the unit sphere.*

Similarly, we next solve for a symmetric equilibrium of the Colonel Blotto game for a class of distribution functions for $n > 3$. The proofs are relegated to the Appendix.

Proposition 3 *Assume that there are* $n > 3$ *objects. Assume that players' values are independently drawn from continuous distributions* $G_1(v_1, ..., v_n)$ *and* $G_2(w_1, ..., w_n)$ *such that densities are of the following form:*

$$g_1(v_1, v_2, v_3) = [v_1 ... v_n]^{\frac{3-n}{n-2}} \widetilde{g}_1\left(\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)\right)$$

and

$$g_2(w_1, w_2, w_3) = [v_1 ... v_n]^{\frac{3-n}{n-2}} \widetilde{g}_2\left(\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)\right),$$

where $\widetilde{g}_1$ *and* $\widetilde{g}_2$ *are measurable functions on* $\mathbb{R}_+$ *such that*

$$\int_0^\infty \widetilde{g}_i(x) x^{\frac{1}{n-1}} dx = \Gamma\left(\frac{n}{n-1}\right)\left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n \left(\frac{n-1}{n-2}\right)^n.$$ [11]

Then, the following is a symmetric equilibrium

$$\beta(v_1, v_2 ..., v_n) = \left(\frac{v_1^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}} B, \frac{v_2^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}} B, ..., \frac{v_n^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}} B\right).$$

Proof. See the Appendix. ■

One distribution example that satisfy the sufficient conditions listed in the Proposition is as follows. Other examples can be easily constructed.

Example 4 *Assume that each* v_i *is independently drawn from Generalized Gamma Distribution with parameters* $(a, d, p) = \left(1, \frac{1}{n-2}, \frac{n-1}{n-2}\right)$ *where general density of this distribution is given by* $\left(\frac{p}{a^d} \frac{x^{d-1} e^{-\left(\frac{x}{a}\right)^p}}{\Gamma\left(\frac{d}{p}\right)}\right)$, $x \geq 0$. *That is, each* v_i *is independently drawn from a*

[11] Again, this makes sure that g_i s are density

distribution function with density

$$f(x) = \left(\frac{n-1}{n-2}\right)\left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right) x^{\frac{3-n}{n-2}} \exp\left(-x^{\frac{n-1}{n-2}}\right) \text{ for } x \geq 0.$$

Then, the joint density will be

$$g(v_1, ..., v_n) = \left(\frac{n-1}{n-2}\right)^n \left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n v_1^{\frac{3-n}{n-2}} ... v_n^{\frac{3-n}{n-2}} \exp\left(-\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)\right)$$

for $v_1, ..., v_n \geq 0$. *That is,* $\widetilde{g}(x) = \left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n \left(\frac{n-1}{n-2}\right)^n \exp(-(x))$ *when* $x \geq 0$. *Note that*

$$\int_0^\infty \exp(-(x))\, x^{\frac{1}{n-1}} dx = \Gamma\left(\frac{n}{n-1}\right)\left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n \left(\frac{n-1}{n-2}\right)^n.$$

3 Efficiency of Auction with Points

As we have argued, auctions with points have been used in practice, as in the course bidding system in business and law schools, for allocating objects to agents when monetary transfers are not possible. Therefore, it is important to understand how this mechanism performs in terms of welfare of agents compare to ordinal mechanisms, where agents report their ordinal rankings over objects and based on these reported preferences an allocation is determined.[12]

To make the comparison, it will be sufficient to consider the so-called Ranking mechanism due to the main result of Akyol (2020). The Ranking mechanism works as follows. Each object is assigned to the agent who ranks that object at a lower spot (that is, who has a higher preference ranking) and if some agents rank the object at the same place, object is given to each agent with equal probability. For example, assume that $n = 3, m = 2$ and first agent's strict ordinal ranking over objects $\{o_1, o_2, o_3\}$ is given by $o_1 \succ o_2 \succ o_3$ and agent 2's strict ordinal ranking is $o_3 \succ o_2 \succ o_1$. Under the Ranking mechanism, o_1 is allocated to agent $1, o_3$ is allocated to agent 2 and o_2 is allocated to each agent with probability $\frac{1}{2}$.

By the main result of Akyol (2020), the Ranking mechanism has strong welfare superiority to other ordinal mechanisms.

Proposition 4 ***Akyol (2020)*** *When each agent draws his valuation vector from an exchangeable distribution function, then every type of agent has a higher interim payoff under the Ranking mechanism compare to any other anonymous, neutral and incentive compatible ordinal mechanism.*

Now, this result at hand, we are going to compare the Blotto mechanism with the Ranking mechanism. First, it is easy to see that when $n = 2$, the outcome of the Colonel

[12] As we have discussed in the Introduction, ordinal mechanisms are predominantly used in many real-life allocation problems.

Blotto game is equivalent to the Ranking mechanism. Therefore, we will restrict attention to the case $n > 2$. When each agent draws his valuation vector from an exchangeable distribution function[13] as is the case in the distributional assumptions in which we obtain equilibrium in closed-form for the Colonel Blotto game, the interim probability of an agent to receive his k^{th} choice under the Ranking mechanism is given by

$$P_k^{rank} = \frac{2n-2k+1}{2}$$

To see this, note that an agent, say agent 1, will obtain his k^{th} choice object with probability 1 if the other agent ranks this object as his j^{th} choice for $j > k$ which happens with probability $\frac{n-k}{n}$ and will obtain his k^{th} choice object with probability $\frac{1}{2}$ if the other agent ranks this object also as his k^{th} choice which happens with probability $\frac{1}{n}$. If the other agent ranks this object as his j^{th} choice for $j < k$, agent 1 can not obtain his k^{th} choice. Hence,

$$P_k^{rank} = 1 \times \frac{n-k}{n} + \frac{1}{2} \times \frac{1}{n} = \frac{2n-2k+1}{2n}$$

We start with the case when $n = 3$.

Assume that agents' values are independently drawn from some distribution as in Proposition 2. Consider an agent with type (v_1, v_2, v_3). Then, this agent's interim payoff under the Blotto mechanism is

$$\Pi^{Blotto}(v_1, v_2, v_3) = \sum_{i=1}^{3} \Pr\left(\frac{v_i^2}{v_1^2+v_2^2+v_3^2} B \geq \frac{W_i^2}{W_1^2+W_2^2+W_3^2} B\right) v_i$$

Then, by the Lemma 1 in Appendix C, we have that

$$\begin{aligned} &\Pi^{Blotto}(v_1, v_2, v_3) \\ = &\sum_{i=1}^{3} \sqrt{\frac{v_i^2}{v_1^2+v_2^2+v_3^2}} v_i \\ = &\sum_{i=1}^{3} \frac{v_i^2}{\sqrt{v_1^2+v_2^2+v_3^2}} \\ = &\sqrt{v_1^2+v_2^2+v_3^2} \end{aligned}$$

and this agent's interim payoff under the Ranking mechanism is

$$\begin{aligned} &\Pi^{rank}(v_1, v_2, v_3) \\ = &\frac{1}{6}(5v_1 + 3v_2 + v_3) \end{aligned}$$

since when $n = 3$, for each $k \in \{1, 2, 3\}$

$$P_k^{rank} = \frac{7-2k}{6}.$$

[13] That is, when valuations are drawn from a distribution that is invariant under permutations of its arguments.

We want to investigate the sign of $\sqrt{v_1^2 + v_2^2 + v_3^2} - \frac{1}{6}(5v_1 + 3v_2 + v_3)$, or, equivalently the sign of $(v_1^2 + v_2^2 + v_3^2) - \frac{(5v_1+3v_2+v_3)^2}{36}$.

$$
\begin{aligned}
& v_1^2 + v_2^2 + v_3^2 - \frac{(5v_1 + 3v_2 + v_3)^2}{36} \\
= \; & \frac{1}{36}\left(11v_1^2 + 27v_2^2 + 35v_3^2 - 30v_1v_2 - 10v_1v_3 - 6v_2v_3\right) \\
= \; & \frac{1}{36}\left((v_1 - 5v_3)^2 + (v_2 - 3v_3)^2 + (3v_1 - 5v_2)^2 + \left(v_1^2 + v_2^2 + v_3^2\right)\right) \\
\geq \; & 0
\end{aligned}
$$

Therefore, we have the following result.

Proposition 5 *Under the distributions considered in Proposition* 2 *, every type of every agent has a higher interim payoff under the Blotto mechanism compare to any anonymous, neutral and incentive compatible ordinal mechanism when* $n = 3$.

Hence, we deduce that the Blotto mechanism is welfare superior to other ordinal mechanisms in a very strong sense.

Same result holds for the case when $n > 3$.

Proposition 6 *Under the distributions considered in Proposition* 3, *any type of any agent has a higher interim payoff under the Blotto mechanism compare to any anonymous, neutral and incentive compatible ordinal mechanism when* $n > 3$.

Proof. See the Appendix. ■

Hence, the Blotto mechanism has a welfare superiority over ordinal mechanisms in a very strong sense—every type has a higher interim payoff under the Blotto mechanism. Therefore, "fake market" environment seems promising to improve upon widely used ordinal mechanisms in terms of welfare.

4 Conclusion

We have considered the problem of allocation multiple objects to agents via an auction with points, which is equivalent to the classical Colonel Blotto game, under incomplete information. Although it is in general hard to come up with closed form expressions when there is multi-dimensional incomplete information, we are able to solve and obtain simple expressions for the equilibrium of this game for a class of value distributions. In addition, we have established a strong welfare superiority of this allocation mechanism to other ordinal mechanisms which are dominantly used in many real-life applications. Hence, creating "fake market" seems a promising method as it is welfare superior to widely-used ordinal allocation methods when prices/monetary transfers cannot be used.

A Proof of Proposition 2

Lemma 1 *Define a random variable* $U \equiv \frac{V_1^2}{V_1^2+V_2^2+V_3^2}$, *where* (V_1, V_2, V_3) *is distributed according to a continuous distribution function that has density of the form* $g(v_1, v_2, v_3) \equiv \widetilde{g}\left(\left(v_1^2 + v_2^2 + v_3^2\right)\right)$ *for some measurable function* $\widetilde{g}$ *on* $\mathbb{R}_+$ *such that*

$$\int_0^\infty \widetilde{g}(x)\, x^{\frac{1}{2}} dx = \frac{4}{\pi}.$$

Then, U *is distributed according to* $F(u) = u^{\frac{1}{2}}$ *with support* $[0, 1]$.

Proof. Let $V = \frac{V_2^2}{V_1^2+V_2^2+V_3^2}$ and $W = V_1^2 + V_2^2 + V_3^2$.

Now,

$$\begin{aligned} V_1^2 &= UW \\ V_2^2 &= VW \\ V_3^2 &= (1 - U - V)\, W, \end{aligned}$$

or,

$$\begin{aligned} V_1 &= U^{\frac{1}{2}} W^{\frac{1}{2}} \\ V_2 &= V^{\frac{1}{2}} W^{\frac{1}{2}} \\ V_3 &= (1 - U - V)^{\frac{1}{2}}\, W^{\frac{1}{2}}. \end{aligned}$$

Then,

$$h(u, v, w) = g\left(u^{\frac{1}{2}} w^{\frac{1}{2}}, v^{\frac{1}{2}} w^{\frac{1}{2}}, (1 - u - v)^{\frac{1}{2}}\, w^{\frac{1}{2}}\right) \det(J),$$

where

$$J = \begin{bmatrix} \frac{1}{2} u^{-\frac{1}{2}} w^{\frac{1}{2}} & 0 & \frac{1}{2} u^{\frac{1}{2}} w^{-\frac{1}{2}} \\ 0 & \frac{1}{2} v^{-\frac{1}{2}} w^{\frac{1}{2}} & \frac{1}{2} v^{\frac{1}{2}} w^{-\frac{1}{2}} \\ -\frac{1}{2}(1 - u - v)^{-\frac{1}{2}}\, w^{\frac{1}{2}} & -\frac{1}{2}(1 - u - v)^{-\frac{1}{2}}\, w^{\frac{1}{2}} & \frac{1}{2}(1 - u - v)^{\frac{1}{2}}\, w^{-\frac{1}{2}} \end{bmatrix}$$

Hence,

$$\begin{aligned} & h(u, v, w) \\ = \;& g\left(u^{\frac{1}{2}} w^{\frac{1}{2}}, v^{\frac{1}{2}} w^{\frac{1}{2}}, (1 - u - v)^{\frac{1}{2}}\, w^{\frac{1}{2}}\right) \left(\frac{1}{8} \frac{w^{\frac{1}{2}}}{u^{\frac{1}{2}} v^{\frac{1}{2}} (1 - u - v)^{\frac{1}{2}}}\right) \\ = \;& \widetilde{g}(w) \left(\frac{1}{8} \frac{w^{\frac{1}{2}}}{u^{\frac{1}{2}} v^{\frac{1}{2}} (1 - u - v)^{\frac{1}{2}}}\right), \end{aligned}$$

where $0 < u, v < 1,\ u + v < 1, w > 0$
Hence,

$$h(u,v,w) = \frac{1}{8}\widetilde{g}(w)\, w^{\frac{1}{2}} \left(\frac{1}{u^{\frac{1}{2}} v^{\frac{1}{2}} (1-u-v)^{\frac{1}{2}}} \right)$$

for $0 < u, v < 1,\ u + v < 1, w > 0$. Now,

$$h(u,v) = \left(\frac{1}{8} \int_0^\infty \widetilde{g}(w)\, w^{\frac{1}{2}} dw \right) \left(\frac{1}{u^{\frac{1}{2}} v^{\frac{1}{2}} (1-u-v)^{\frac{1}{2}}} \right)$$

for $0 < u, v < 1,\ u + v < 1$. Hence,

$$h(u,v) = \frac{1}{2\pi} \left(\frac{1}{u^{\frac{1}{2}} v^{\frac{1}{2}} (1-u-v)^{\frac{1}{2}}} \right)$$

for $0 < u, v < 1,\ u + v < 1$. Then,

$$\begin{aligned} h(u) &= \int_0^{1-u} \frac{1}{2\pi} \left(\frac{1}{u^{\frac{1}{2}} v^{\frac{1}{2}} (1-u-v)^{\frac{1}{2}}} \right) dv \\ &= \frac{1}{2} u^{-\frac{1}{2}}, \end{aligned}$$

for $0 < u < 1$. Hence,

$$H(u) = u^{\frac{1}{2}},\ 0 \le u \le 1,$$

which proves the lemma. ■

Proof. (of Proposition 2**)** Assume that Player 2 follows the given strategy. Assume that Player 1 has values (v_1, v_2, v_3). Then, he solves

$$\begin{aligned} &\max_{0 \le b_1, b_2 \le B} \Pr\left(b_1 \ge \frac{w_1^2}{w_1^2 + w_2^2 + w_3^2} \right) v_1 + \Pr\left(b_2 \ge \frac{w_2^2}{w_1^2 + w_2^2 + w_3^2} \right) v_2 + \\ &+ \Pr\left(B - b_1 - b_2 \ge \frac{w_3^2}{w_1^2 + w_2^2 + w_3^2} \right) v_3 \end{aligned}$$

Then, by above Lemma, the problem becomes

$$\max_{0 \le b_1, b_2 \le B} b_1^{\frac{1}{2}} v_1 + b_2^{\frac{1}{2}} v_2 + (B - b_1 - b_2)^{\frac{1}{2}} v_3$$

FOC:

$$b_1^{-\frac{1}{2}} v_1 = (B - b_1 - b_2)^{-\frac{1}{2}} v_3$$

and

$$b_2^{-\frac{1}{2}} v_2 = (B - b_1 - b_2)^{-\frac{1}{2}} v_3.$$

Hence,

$$b_2 = b_1 \left(\frac{v_2^2}{v_1^2} \right)$$

and

$$B - b_1 - b_2 = b_1 \left(\frac{v_3^2}{v_1^2} \right).$$

Thus,

$$\begin{aligned} B &= b_1 + b_2 + (B - b_1 - b_2) \\ &= b_1 \left(1 + \left(\frac{v_2^2}{v_1^2} \right) + \left(\frac{v_3^2}{v_1^2} \right) \right) \\ &= b_1 \left(\frac{v_1^2 + v_2^2 + v_3^2}{v_1^2} \right). \end{aligned}$$

Hence,

$$b_1 = B \frac{v_1^2}{v_1^2 + v_2^2 + v_3^2}$$

and

$$b_2 = B \frac{v_2^2}{v_1^2 + v_2^2 + v_3^2}.$$

Hence, $\beta(.)$ is a best response to itself , giving the desired result. ■

B Proof of Proposition 3

Lemma 2 *Define a random variable* $U_1 \equiv \frac{V_1^{\frac{n-1}{n-2}}}{V_1^{\frac{n-1}{n-2}} + ... + V_n^{\frac{n-1}{n-2}}}$, *where* $(V_1, ..., V_n)$ *is distributed according to a continuous distribution function that has density of the form* $g(v_1, ..., v_n) \equiv [v_1 ... v_n]^{\frac{3-n}{n-2}} \widetilde{g} \left(\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}} \right) \right)$ *for some measurable function* $\widetilde{g}$ *on* $\mathbb{R}_+$ *such that*

$$\int_0^\infty \widetilde{g}(x) x^{\frac{1}{n-1}} dx = \left(\frac{\Gamma\left(\frac{n}{n-1}\right)}{\Gamma^n\left(\frac{1}{n-1}\right)} \right) \left(\frac{n-1}{n-2} \right)^n.$$

Then, U_1 *is distributed according to* $F(u) = u^{\frac{1}{n-1}}$ *with support* $[0, 1]$.

Proof. Let for any $i \in \{2, ..., n-1\}$, define $U_i \equiv \frac{V_i^{\frac{n-1}{n-2}}}{V_1^{\frac{n-1}{n-2}} + ... + V_n^{\frac{n-1}{n-2}}}$ and let $W = V_1^{\frac{n-1}{n-2}} + ... + V_n^{\frac{n-1}{n-2}}$.Now,

$$\begin{aligned} V_1^{\frac{n-1}{n-2}} &= U_1 W \\ &... \\ V_{n-1}^{\frac{n-1}{n-2}} &= U_{n-1} W \\ V_n^{\frac{n-1}{n-2}} &= (1 - U_1 - ... - U_{n-1}) W, \end{aligned}$$

or,

$$\begin{aligned} V_1 &= U_1^{\frac{n-2}{n-1}} W^{\frac{n-2}{n-1}} \\ &... \\ V_{n-1} &= U_{n-1}^{\frac{n-2}{n-1}} W^{\frac{n-2}{n-1}} \\ V_n &= (1 - U_1 - ... - U_{n-1})^{\frac{n-2}{n-1}} W^{\frac{n-2}{n-1}}. \end{aligned}$$

Then,

$$\begin{aligned} & h(u_1, ..., u_{n-1}, w) \\ = \; & g\left(u_1^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}}, ..., u_{n-1}^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}}, (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}}\right) \det(J) \end{aligned}$$

Now,[14]

$$\det(J) = \left(\frac{n-2}{n-1}\right)^n \frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}} ... u_{n-1}^{\frac{1}{n-1}} (1 - u_1 - ... - u_{n-1})^{\frac{1}{n-1}}}.$$

Hence,

$$\begin{aligned} & h(u_1, ..., u_{n-1}, w) \\ = \; & g\left(u_1^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}}, ..., u_{n-1}^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}}, (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}}\right) * \\ & \left(\left(\frac{n-2}{n-1}\right)^n \frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}} ... u_{n-1}^{\frac{1}{n-1}} (1 - u_1 - ... - u_{n-1})^{\frac{1}{n-1}}}\right) \\ = \; & [u_1 ... u_{n-1} (1 - u_1 - ... - u_{n-1})]^{\frac{3-n}{n-1}} \left(w^{n\frac{3-n}{n-1}}\right) \widetilde{g}(w) * \\ & \left(\left(\frac{n-2}{n-1}\right)^n \frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}} ... u_{n-1}^{\frac{1}{n-1}} (1 - u_1 - ... - u_{n-1})^{\frac{1}{n-1}}}\right) \\ = \; & \left(\left(\frac{n-2}{n-1}\right)^n\right) [u_1 ... u_{n-1} (1 - u_1 - ... - u_{n-1})]^{\frac{2-n}{n-1}} \widetilde{g}(w) w^{\frac{1}{n-1}}. \end{aligned}$$

[14] See Lemma for a proof of this

Then,

$$\begin{aligned} h\left(u_1,...,u_{n-1}\right) &= \left(\int_0^\infty \widetilde{g}\left(w\right) w^{\frac{1}{n-1}} dw\right) \left(\left(\frac{n-2}{n-1}\right)^n\right) \left[u_1 ... u_{n-1}\left(1-u_1-...-u_{n-1}\right)\right]^{\frac{2-n}{n-1}} \\ &= \left(\frac{\Gamma\left(\frac{n}{n-1}\right)}{\Gamma^n\left(\frac{1}{n-1}\right)}\right) \left[u_1 ... u_{n-1}\left(1-u_1-...-u_{n-1}\right)\right]^{\frac{2-n}{n-1}} \end{aligned}$$

for $0 < u_1 ..., u_{n-1} < 1$, $u_1 + ... + u_{n-1} < 1$. But, this is a Dirichlet Distribution with parameters $\left(\frac{1}{n-1}, \frac{1}{n-1}, ..., \frac{1}{n-1}\right)$ and hence

$$H\left(u\right) = u^{\frac{1}{n-1}},\ 0 \leq u \leq 1,$$

which proves the lemma. ■

Proof. (of the Proposition 3**)** Assume that Player 2 follows the given strategy. Assume that Player 1 has values $\left(v_1, ..., v_n\right)$. Then, he solves

$$\max_{\substack{0 \leq b_1, ... b_{n-1} \leq B \\ b_1 + ... + b_{n-1} \leq B}} \Pr\left(b_1 \geq X_1\right) v_1 + ... + \Pr\left(B - b_1 - ... - b_{n-1} \geq X_n\right) v_n$$

where for all $i \in \{1, ..., n\}$

$$X_i = \frac{w_i^{\frac{n-1}{n-2}}}{w_1^{\frac{n-1}{n-2}} + w_2^{\frac{n-1}{n-2}} + ... + w_n^{\frac{n-1}{n-2}}}.$$

Then, by above Lemma, the problem becomes

$$\max_{\substack{0 \leq b_1, ... b_{n-1} \leq B \\ b_1 + ... + b_{n-1} \leq B}} b_1^{\frac{1}{n-1}} v_1 + ... + \left(B - b_1 - ... - b_{n-1}\right)^{\frac{1}{n-1}} v_n$$

FOC:

$$\begin{aligned} b_1^{\frac{2-n}{n-1}} v_1 &= \left(B - b_1 - ... - b_{n-1}\right)^{\frac{2-n}{n-1}} v_n \\ &... \\ b_{n-1}^{\frac{2-n}{n-1}} v_{n-1} &= \left(B - b_1 - ... - b_{n-1}\right)^{\frac{2-n}{n-1}} v_n. \end{aligned}$$

Hence,

$$b_i = b_1 \left(\frac{v_i^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}}}\right)$$

for all $i \in \{2,..,n-1\}$ Thus,

$$\begin{aligned} B &= b_1 \left(1 + \left(\frac{v_2}{v_1}\right)^{\frac{n-1}{n-2}} + ... + \left(\frac{v_n}{v_1}\right)^{\frac{n-1}{n-2}}\right) \\ &= b_1 \left(\frac{v_1^{\frac{n-1}{n-2}} + v_2^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}}}\right). \end{aligned}$$

Hence,

$$b_1 = B \frac{v_1^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + v_2^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}}$$

and similarly others. ■

Lemma 3 (Determinant of Jacobian)

$$\det(J) = \left(\frac{n-2}{n-1}\right)^n \frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}} ... u_{n-1}^{\frac{1}{n-1}} (1 - u_1 - ... - u_{n-1})^{\frac{1}{n-1}}}.$$

Proof. Now,

$$J = \begin{bmatrix} \frac{n-2}{n-1} u_1^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & 0 & 0 & . & 0 & \frac{n-2}{n-1} u_1^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \\ 0 & \frac{n-2}{n-1} u_2^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & 0 & . & 0 & \frac{n-2}{n-1} u_2^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & \frac{n-2}{n-1} u_{n-1}^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & \frac{n-2}{n-1} u_{n-1}^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \\ -\frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & -\frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & . & . & -\frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & \frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \end{bmatrix}$$

, where $u_n = 1 - u_1 - ... - u_{n-1}$. Denoting $J = (a_{i,j})_{i,j=1}^n$

$$\det(J) = \sum_{i=1}^{n} (-1)^{n+i} a_{n,i} M_{n,i},$$

where $M_{n,i}$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained from J by deleting n-th row and $i-th$ column. Now, for all $1 \leq i \leq n-1$

$$M_{n,i} = (-1)^{n+i-1} \left(\frac{n-2}{n-1}\right)^{n-1} \left(u_i^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1}\right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} u_j\right)^{\frac{n-2}{n-1}-1} w^{\frac{(n-2)^2}{n-1}}$$

and

$$M_{n,n} = \left(\frac{n-2}{n-1}\right)^{n-1} \left(\prod_{j=1}^{n-1} u_j\right)^{\frac{n-2}{n-1}-1} w^{n-2}.$$

Hence,

$$
\begin{aligned}
& \det(J) \\
= \; & \sum_{i=1}^{n} (-1)^{n+i} a_{n,i} M_{n,i} \\
= \; & \sum_{i=1}^{n-1} (-1)^{n+i} \left[- \left(\frac{n-2}{n-1} (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} \right) \right] * \\
& \left[(-1)^{n+i-1} \left(\frac{n-2}{n-1} \right)^{n-1} u_i^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} u_j \right)^{\frac{n-2}{n-1}-1} w^{\frac{(n-2)^2}{n-1}} \right] + \\
& + \left[\left(\frac{n-2}{n-1} \right) (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \right] \left[\left(\frac{n-2}{n-1} \right)^{n-1} \left(\prod_{j=1}^{n-1} u_j \right)^{\frac{n-2}{n-1}-1} w^{n-2} \right] \\
= \; & \left[\left(\frac{n-2}{n-1} \right)^{n} (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}-1} (u_1 ... u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1} \right] * \\
& \left[w^{\frac{n-2}{n-1}+\frac{(n-2)^2}{n-1}} (u_1 + ... + u_{n-1}) + w^{n-2} (1 - u_1 - ... - u_{n-1}) \right] \\
= \; & \left[\left(\frac{n-2}{n-1} \right)^{n} (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}-1} (u_1 ... u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1} \right] * \\
& \left[w^{(n-2)} (u_1 + ... + u_{n-1}) + w^{n-2} (1 - u_1 - ... - u_{n-1}) \right] \\
= \; & \left[\left(\frac{n-2}{n-1} \right)^{n} (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}-1} (u_1 ... u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1} \right] \left[w^{(n-2)} \right] \\
= \; & \left(\frac{n-2}{n-1} \right)^{n} (1 - u_1 - ... - u_{n-1})^{\frac{n-2}{n-1}-1} (u_1 ... u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1+(n-2)} \\
= \; & \left(\frac{n-2}{n-1} \right)^{n} \frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}} ... u_{n-1}^{\frac{1}{n-1}} (1 - u_1 - ... - u_{n-1})^{\frac{1}{n-1}}}.
\end{aligned}
$$

■

C Proof of Proposition 6

Proof. Now, consider a bidder with type $(v_1, ..., v_n)$. Then, his expected payoff

$$
\Pi^{Blotto}(v_1, ..., v_n) = \sum_{i=1}^{n} \Pr \left(\frac{v_1^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}} \geq \frac{W_i^{\frac{n-1}{n-2}}}{W_1^{\frac{n-1}{n-2}} + ... + W_n^{\frac{n-1}{n-2}}} \right) v_i
$$

Then, by the lemma below, we have that

$$
\begin{aligned}
& \Pi^{Blotto}(v_1,..,v_n) \\
= & \sum_{i=1}^{n}\left(\frac{v_i^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}}\right)^{\frac{1}{n-1}} v_i \\
= & \sum_{i=1}^{n}\frac{v_i^{\frac{n-1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{1}{n-1}}} \\
= & \left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi^{rank}(v_1,...,v_n) &= \sum_{i=1}^{n} P_i v_i \\
&= \sum_{i=1}^{n} v_i\left(\frac{1}{2n}+1-\frac{i}{n}\right) \\
&= \left(\frac{1}{2n}+1\right)\left(\sum_{i=1}^{n} v_i\right)-\frac{1}{n}\left(\sum_{i=1}^{n} i v_i\right) \\
&= \frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n} v_i\right)-2\left(\sum_{i=1}^{n} i v_i\right)\right)
\end{aligned}
$$

Now,

$$
\begin{aligned}
& d(v_1,..,v_n) \\
= & \Pi^{Blotto}(v_1,..,v_n)-\Pi^{rank}(v_1,...,v_n) \\
= & \left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n} v_i\right)-2\left(\sum_{i=1}^{n} i v_i\right)\right)
\end{aligned}
$$

We claim that $d(v_1,..,v_n)\geq 0$ for all $\mathbf{v}=(v_1,..,v_n)\in[\underline{v},\overline{v}]^n$ for any $\overline{v}>\underline{v}\geq 0$ and $v_1\geq v_2\geq...\geq v_n$. To do this we will show that

$$
0\leq\left[\min_{v_1\geq v_2\geq...\geq v_n\geq\underline{v}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n} v_i\right)-2\left(\sum_{i=1}^{n} i v_i\right)\right)\right]
$$

Note that

$$\begin{aligned}
& \left[\min_{\overline{v}\geq v_1\geq v_2\geq\ldots\geq v_n\geq \underline{v}}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n}v_i\right)-2\left(\sum_{i=1}^{n}iv_i\right)\right)\right] \\
\geq & \left[\min_{v_1\geq v_2\geq\ldots\geq v_n\geq 0}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n}v_i\right)-2\left(\sum_{i=1}^{n}iv_i\right)\right)\right] \\
\geq & \left[\min_{v_1,v_2,\ldots v_n\geq 0}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n}v_i\right)-2\left(\sum_{i=1}^{n}iv_i\right)\right)\right]
\end{aligned}$$

Actually, it is easy to see that last two problems are equivalent, but weak inequality is sufficient for our purpose. Now, to show the desired result we will concentrate on the last minimization problem. First, observe that

$$\begin{aligned}
& \frac{\partial}{\partial v_i}d\left(v_1,..,v_n\right) \\
= & v_i^{\frac{1}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}}-\frac{2n+1}{2n}+\frac{i}{n} \\
= & \frac{v_i^{\frac{1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{\frac{1}{n-1}}}-\frac{2n+1}{2n}+\frac{i}{n}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial v_i\partial v_j}d\left(v_1,..,v_n\right) &= -\frac{1}{n-2}v_j^{\frac{1}{n-2}}v_i^{\frac{1}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{n}{n-1}} \\
&\leq 0
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2}{\partial v_i^2}d\left(v_1,..,v_n\right) \\
= & \frac{1}{n-2}v_i^{\frac{3-n}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}}-\frac{1}{n-2}v_i^{\frac{2}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{n}{n-1}} \\
= & \frac{1}{n-2}v_i^{\frac{3-n}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{n}{n-1}}\left[\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)-v_i^{\frac{n-1}{n-2}}\right] \\
\geq & 0
\end{aligned}$$

For a given $v_{-i}\neq\mathbf{0}$, $\frac{\partial}{\partial v_i}\left(d\left(v_1,..,v_n\right)\right)=\frac{v_i^{\frac{1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{\frac{1}{n-1}}}-\frac{2n+1}{2n}+\frac{i}{n}<0$ for $v_i=0$ since $i\leq n$, then the minimizing v_i for a given $v_{-i}\neq\mathbf{0}$ is such that it solves

$$\frac{v_i^{\frac{1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}}+\ldots+v_n^{\frac{n-1}{n-2}}\right)^{\frac{1}{n-1}}}-\frac{2n+1}{2n}+\frac{i}{n}=0$$

Furthermore, for $v_{-i} = \mathbf{0}$, the minimizing $v_i = 0$ since for $v_{-i} = \mathbf{0}$, minimization problem becomes

$$\min_{v_i \geq 0} v_i - \frac{1}{2n}\left[(2n+1) - 2i\right] v_i$$

or

$$\min_{v_i \geq 0} v_i \left(1 - \frac{2n+1-2i}{2n}\right)$$

$$\min_{v_i \geq 0} v_i \left(\frac{2i-1}{2n}\right)$$

Since $\left(\frac{2i-1}{2n}\right) > 0$, minimizing $v_i = 0$.

Given these suppose that minimizing $\mathbf{v} \neq \mathbf{0}$. That is, there is some j such that $v_j \neq 0$. But then, by above observation $v_i \neq 0$ for all $i \neq j$.

Now, then each i satisfies

$$v_i^{\frac{1}{n-2}} \left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}} - \frac{2n+1}{2n} + \frac{i}{n} = 0$$

or

$$v_i^{\frac{1}{n-2}} \left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}} = \frac{2n+1}{2n} - \frac{i}{n}$$

$$\frac{v_i^{\frac{1}{n-2}}}{v_j^{\frac{1}{n-2}}} = \frac{\frac{2n+1}{2n} - \frac{i}{n}}{\frac{2n+1}{2n} - \frac{j}{n}}$$

$$\frac{v_i}{v_j} = \left(\frac{\frac{2n+1}{2n} - \frac{i}{n}}{\frac{2n+1}{2n} - \frac{j}{n}}\right)^{n-2}$$

Hence,

$$v_i = v_1 \left(\frac{\frac{2n+1}{2n} - \frac{i}{n}}{\frac{2n+1}{2n} - \frac{1}{n}}\right)^{n-2} \quad ,i > 1$$

or

$$v_i = v_1 \left(\frac{2n+1-2i}{2n-1}\right)^{n-2} \quad ,i > 1$$

Then, the objective function $d(.,..,.)$ becomes:

$$\begin{aligned}
& \left(\sum_{i=1}^{n} v_1^{\frac{n-1}{n-2}} \left(\frac{2n-2i+1}{2n-1}\right)^{n-1}\right)^{\frac{n-2}{n-1}} \\
& -\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n} v_1 \left(\frac{2n-2i+1}{2n-1}\right)^{n-2}\right) - 2\left(\sum_{i=1}^{n} v_1 i \left(\frac{2n-2i+1}{2n-1}\right)^{n-2}\right)\right) \\
= \; & \frac{v_1}{(2n-1)^{n-2}}\left[\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}} - \frac{1}{2n}\left(\sum_{i=1}^{n}(2n+1-2i)(2n-2i+1)^{n-2}\right)\right] \\
= \; & \frac{v_1}{(2n-1)^{n-2}}\left[\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}} - \frac{1}{2n}\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)\right] \\
= \; & \frac{v_1}{(2n-1)^{n-2}(2n)}\left[(2n)\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}} - \left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)\right] \\
= \; & \left[\frac{v_1}{(2n-1)^{n-2}(2n)}\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}}\right]\left[2n - \left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{1}{n-1}}\right]
\end{aligned}$$

Note that the term in the first bracket is positive. ■

Claim 1 $\left[2n - \left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{1}{n-1}}\right] > 0$ *for* $n \geq 3$

Proof. We want to show

$$(2n)^{n-1} > \sum_{i=1}^{n}(2n-2i+1)^{n-1}$$

or

$$n^{n-1} > \sum_{i=1}^{n}\left(\frac{2n-2i+1}{2}\right)^{n-1} = \sum_{i=1}^{n}\left(n-i+\frac{1}{2}\right)^{n-1}$$

Now, for each $i \in \{1,...,n\}$

$$\left(n-i+\frac{1}{2}\right)^{n-1} < \int_{n-i}^{n-i+1} t^{n-1}dt$$

since

$$\int_{n-i}^{n-i+1} t^{n-1}dt = \int_{0}^{\frac{1}{2}} \left[(n-i+u)^{n-1} + (n-i+1-u)^{n-1}\right] du$$

$$> \int_{0}^{\frac{1}{2}} 2\left(n-i+\frac{1}{2}\right)^{n-1} du$$

$$= \left(n-i+\frac{1}{2}\right)^{n-1}$$

where strict inequality is due to the fact that $f(t) = t^{n-1}$ is a strictly convex function. Thus,

$$\sum_{i=1}^{n} \left(n-i+\frac{1}{2}\right)^{n-1} < \sum_{i=1}^{n} \int_{n-i}^{n-i+1} t^{n-1}dt$$

$$= \int_{0}^{n} t^{n-1}dt$$

$$= n^{n-1}$$

which is what we wanted to show. Thus, the value objective function is positive when $v_1 > 0$ but note that the value of objective function is 0 when $\mathbf{v} = \mathbf{0}$. Hence, the minimum is achieved at $v_i = v_j = 0$ for all i, j giving that for all $\mathbf{v}$

$$d(v_1, .., v_n) \geq 0$$

■

References

[1] Adamo, T., & Matros, A. (2009). A Blotto game with incomplete information. *Economics Letters*, 105(1), 100-102.

[2] Akyol, E. (2014). Essays On Mechanism Design Without Transfers. Penn State University.

[3] Akyol, E. (2020). Allocation without Transfers: A Welfare Maximizing Mechanism. Working paper.
Available at https://www.dropbox.com/s/vmgaiyn2k4fnyy0/notransfers.pdf?dl=0.

[4] Borel, E. (1921). La Theorie de Jeu et les Equations Integrales a Noyan Symetrique. *Comptes Rendus de l'Academie des Sciences,* 173,1304–1308.

[5] Che, Y. K., & Kojima, F. (2010). Asymptotic equivalence of probabilistic serial and random priority mechanisms. *Econometrica*, 78(5), 1625-1672.

[6] Ewerhart, C., and Kovenock, D., A Class of N-Player Colonel Blotto Games with Multidimensional Private Information (November 14, 2019). University of Zurich, Department of Economics, Working Paper No. 336, Available at SSRN: https://ssrn.com/abstract=3487546 or http://dx.doi.org/10.2139/ssrn.3487546.

[7] Gross, O. A., and Wagner, R. A. (1950). A Continuous Colonel Blotto Game. *Santa Monica, CA: RAND Corporation*, http://www.rand.org/pubs/research_memoranda/RM408

[8] Hortala-Vallve, R., & Llorente-Saguer, A. (2012). Pure strategy Nash equilibria in non-zero sum colonel Blotto games. *International Journal of Game Theory*, 41(2), 331-343.

[9] Kovenock, D., & Roberson, B. (2011). A Blotto game with multi-dimensional incomplete information. *Economics Letters*, 113(3), 273-275.

[10] Roberson, B. (2006). The Colonel Blotto Game. *Economic Theory*, 1-24.

[11] Sönmez, T., & Ünver, M. U. (2010). Course bidding at business schools. *International Economic Review*, 51(1), 99-123..

[12] Thomas, C. (2018) N-Dimensional Blotto Games with Asymmetric Battlefield Values, *Economic Theory*, Volume 65 (3), 509-544